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Vol. 244 JULY, 1947 No. 1

THE FUTURE OF NUCLEAR PHYSICS.

Address made by Dr. Enrico Fermi, Physicist, Institute of Nuclear Studies of the University of Chicago, on the occasion of the award to him of the Franklin Medal of The Franklin Institute, April 16, 1947.

Mr. Chairman, Ladies and Gentlemen:

It is a great pleasure to be able to talk before you and to express to The Franklin Institute my deep appreciation for the honor that they have conferred on me.

In discussing with you the future of Nuclear Physics, I will not spend much time tonight in illustrating the possibilities of this branch of science in the field of the development of atomic energy for peace or war purposes, but I would like to stress the future of this field of physics from the point of view of what it might teach us of the intimate structure of matter. The history of events, both recent and remote, has taught us that not only scientific but also basic technological progress is closely connected with our understanding of the mechanism of natural phenomena.

Although nuclear physics already has yielded very impressive practical results, and may yield more in the not too distant future, it would be undoubtedly narrow-sighted to take the point of view that our only partial understanding of the phenomena of the atomic nucleus is adequate.

The dynamical theory of the planetary system reached its perfection with Newton's discovery of the law of force between the sun and the planets. Unfortunately, no Newton has yet appeared in the field of nuclear physics to teach us what are the laws of forces between neutrons and protons in a nucleus. We have only some very few hints as to the nature of these forces; namely, that they are of exceedingly short range and are appreciable only when the particles are at distances of the order

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(Note—The Franklin Institute is not responsible for the statements and opinions advanced by contributors in the Journal.)

of one-tenth of a millionth of a millionth of an inch. We know also the order of the magnitude of their intensity, but there our information practically ends! We do not know, for instance, by what law of the distances they decrease, nor their dependence upon the relative positions of the particles between which such forces act.

A very significant attempt to interpret and compute the properties of nuclear forces was made shortly before the war, with the Mesotron Theory proposed by Yukawa. From the fact that nuclear forces are of short range, he could predict the existence of particles of mass intermediate between those of the electron and of the proton. The fact that such particles were actually found in the cosmic radiation constituted a very encouraging success for his views. Unfortunately, subsequent work has somewhat dampened the hopes founded on this theory and all attempts to put it in precise form so far have proved unworkable.

Theory is usually rather helpless in attacking a thoroughly new problem unless it is supported by experimentation. For this reason, there are now great hopes of further progress. New experiments will become possible with the development of giant cyclotrons, like the one recently operated at Berkeley, and other large accelerating machines, which are now being planned and developed in several Institutions. These machines will permit to study in the laboratory particles of energies approaching those of cosmic rays. It seems justifiable to expect that their investigation in the laboratory may offer valuable leads for the exploration of nuclear properties. Many physicists hope that it may even be possible to produce artificial mesotrons and to demonstrate directly their connection with nuclear forces postulated by Yukawa.

The overall problem of the nucleus, however, will not be solved by the knowledge of the forces alone. Many nuclei are of extreme complexity and contain hundreds of neutrons and protons closely packed together, so that even if the laws of their dynamics were understood, their application to such a complex system would present a formidable mathematical problem.

Its complexity is such that the hopes of finding exact solutions by the conventional methods of analysis are exceedingly small and it appears more probable that numerical methods will have to be used. I would like to point out the importance that electronic computing machines will have in this respect. The Eniac, an early model of this type of machines, has been developed in Philadelphia during the war and is now operating successfully. Some promising results in its application to problems of nuclear physics already have been obtained. But the men who are responsible for the Eniac development are not resting on their present achievements and work is actively going on to build what may properly be called an electronic mathematical brain. I put great hopes in the help that the physicists will be able to derive by these very promising developments.

What I have outlined represents a very extensive program that will probably take many years and exceedingly hard work to be carried through.

One might expect that, at the same time, work will go on actively in the field of the practical applications of atomic energy. The hope has been expressed and does not seem unreasonable that we might witness in the next ten or twenty years the development of more and more extensive practical use of atomic power plants.

We may see radioactive tracers produced in the atomic piles become a more and more important tool for biology and chemistry.

We may also witness new practical inventions in the field of atomic energy, of which we now have no conception. Such a program of technical development will undoubtedly command the largest share of public attention. Beside such large size enterprises, the smaller scale efforts in the field of pure science may well appear less significant. It is my hope, however, that this shall not lead to a lessening of the effort to uncover new basic principles. Since this, I firmly believe, is the true and ultimate goal of science.

2000 Horsepower Gas Turbine Generator Set. THOMAS J. PUTZ. (*Blast Furnace and Steel Plant*, Vol. 34, No. 12.)—In a paper presented before the Annual Meeting of the American Society of Mechanical Engineers there was described an experimental 2000-horsepower gas-turbine generator set that weighs only 19 pounds per horsepower and starts in 1½ minutes or less developed for industrial, central station, marine, and locomotive applications. Occupying but ¼ cubic foot per horsepower, its light weight, compactness, and quick starting and loading response make possible among other things a gas-turbine-electric locomotive less than half the length and two-thirds the weight of its equivalent in a diesel-electric.

Burning No. 6 (bunker "C") fuel oil to compensate economically for an expected thermal efficiency of its simple open cycle of twenty per cent. at full load, the gas turbine generator set operates at a top temperature of 1350° F.—the maximum practical temperature using currently available materials without resort to cooling. The simplest type of open cycle, consisting of only a compressor, combustor and turbine is employed in order that operating experience on these major components may be most quickly acquired, and the useful output is absorbed by a double-armature D.-C. traction-type generator connected to the turbine through a single reduction gear. Tests to date indicate smooth mechanical performance.

The possible applications of such a simple open cycle gas turbine are numerous. As a locomotive prime mover it can be built in large powers within a single cab and would require no cooling water. Four of the present units could be installed in a standard cab.

For marine auxiliary applications it may find a place as a standby power unit and as power booster, and with regenerators and inter-coolers added to raise thermal efficiency it may serve as the main drive. For large ships, however, the closed cycle may be adopted to reduce weight and space.

In the central station field the simple open cycle gas turbine lends itself particularly on the ends of transmission lines to use as a standby unit for peak service. The expense of banked boilers and spinning reserves required of central station steam plants could be reduced by the substitution of low cost, quick starting gas turbines which could be strategically located to cover system peak loads.

Other applications such as portable power plants in the gas fields, power units for gas transmission lines, and power plants for factories where circumstances make it advantageous are all attractive possibilities.

In operation air is taken in by the compressor through the inlet metering nozzle and silencer and compressed to pressures of 30 to 75 p.s.i. absolute, depending on the load carried. Fuel oil is mixed with the compressed air and burned in the combustors, the amount of fuel burned being controlled to limit the temperature of the gases to between 700° F. and 1350° F. at the combustion chamber outlets. The hot gases are expanded through the power-producing turbine and the resultant exhaust gases pass through a diffuser, elbow, and silencer to the atmosphere.

The turbine develops approximately 6000 horsepower, of which 4000 horsepower is used to drive the compressor. The remaining 2000 horsepower is the useful output delivered to the d.-c. generator. The full load speed of the turbine and compressor is 9200 r.p.m. while the generator speed is 1200 r.p.m.

R. H. OPPERMANN.

SYNTHESIS OF PENICILLIN.*

BY

SIR ROBERT ROBINSON, F.R.S., Hon. LL.D., Hon. D.Sc.,

Wayneflete Professor of Chemistry, Oxford University, England.

The biological importance of penicillin focused attention on the study of its chemistry from about 1941, but there were several good reasons for accelerating the tempo of the investigation during the war.

The mould is not an economical producing unit and it is not under control; even now the cost of the antibiotic is much greater than that of the great majority of synthetic pharmaceuticals. Hence laboratory synthesis and factory manufacture was hoped for, and that not merely in the interest of cheapness but also because it was expected that a host of analogues would be accessible. Some of these might be of outstanding therapeutic interest.

Impressive scientific man-power was mobilized for the attack and the outcome was the exploration of a vast new territory; the results will soon be described in a monograph of about a million words which will be published by the Princeton University Press.

The research began in Oxford, but was quickly extended until it became a considerable collaborative effort in the United States and United Kingdom, in which the chief pharmaceutical firms in both countries joined with Government Institutions and University laboratories. Every resource of classical organic chemistry and microchemistry was called upon and the finishing touches were given by the sharp tools of physical methods, such as X-ray crystal analysis and infra-red absorption spectra.

The molecular structure of penicillin, I should say, the penicillins, for there are several substances of the class, was determined and found to be unique among natural products.

Quite early a synthesis was discovered but it was very unsuccessful from the point of view of yield and it was even uncertain that the product really was penicillin. It would not have been discovered at all, if the great antibiotic potency had not been available as a test for the smallest traces of the substance. First it was noticed that the small activities obtained were destroyed by a specific enzyme. Then it was shown that the synthetic material showed the same ratios of activities against seven different bacteria as penicillin; then that synthetic isotopic penicillin was inseparable from penicillin; finally, as a *tour de force*, enough of the synthetic penicillin was made to enable its isolation in small amount to be effected. The synthesis of penicillin has thus been demonstrated but the yield is only 0.1 per cent., one part in a thousand of what might have been obtained. All attempts to improve on this miserable result have been completely unsuccessful and it appears likely that we must wait for some new method or device, which will perhaps arise from some seemingly unrelated investigation.

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* Talk presented at the Medal Day Meeting at The Franklin Institute, April 16, 1947.

The incident shows how much a team can accomplish and also the limitations of planned and concerted work towards a defined objective. There is no doubt but that the concentrated effort led to a rapid solution of all the problems, in so far as they could be resolved by applications of ascertained knowledge. But a practicable synthesis seems to await a discovery, and one does not know where to look for it. Hence the progress of science requires, not only planned collaborative work, but also the bow released at a venture, in other words independent, undirected, even random investigation. Naturally one does not wait for discoveries to happen. As Benjamin Franklin said "Diligence is the mother of luck," and surely discovery is the kind of luck that is peculiarly the reward of diligence.

There is no antithesis, as is so often suggested, in the philosophies of the planner and the free scientist. Both must be given full scope in an intelligent society.

THE EFFECT OF THE WAR UPON PRODUCTION STANDARDS AND THE OUTLOOK FOR THE FUTURE.*

BY

RALPH KELLY,

President, Baldwin Locomotive Works, Philadelphia.

Your flattering invitation to speak before this distinguished and thoughtful gathering upon the technical subject of Production Standards indicates a more than average interest in this important subject. That we have a number of ladies in our audience is also a high compliment, and important, because of the large number of ladies in industry sometimes termed "females" in the records. It has been my experience that ladies when employed for their type of work are universally steadier and better employees than men.

Probably the most important problem facing management today is that of attaining a productive efficiency which will permit the payment of the high rates now received by American workmen and which will, at the same time, enable goods to be sold at prices which are not prohibitive for the average purchaser.

The uniform "across the board" increases which have been granted as national patterns for all industry during the past two years make no economic sense. The only source of higher real wages is increased output per man hour of work. We have ignored this fundamental fact of wage economics in the past few years and granted very sizable advance payments against future increases in production which have not been realized.

After V-J Day, American workers were told by people in high places that their incomes could be maintained at inflated war time levels with a reduction in working hours. Something for nothing was promised. They were told that higher wages would not increase their cost of living.

The price increases in the past year have demonstrated the fallacies of this politically inspired propaganda and we have all rediscovered the ancient truth that unless there is a corresponding increase in production, and "across the board" increase in pay simply results in an equivalent across the board increase in the cost of goods that are purchased.

To make matters worse, there was a marked decline in productivity during the war years which was very much in evidence upon reconversion to peacetime products. Henry Ford testified about a year ago that on an overall basis the Ford Motor Company was getting about 34 per cent less output per direct labor man hour for comparable products and processes than it obtained in 1941. His survey showed the man hours required to manufacture comparable vehicles increased from 87 to 128 from the 1941 period to the 1945 period. General Motors estimates that the present worker productive effort is 21 per cent less than that of the prewar period.

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* Presented under the Charles Day Lecture Foundation of The Franklin Institute of Pennsylvania, given in the Hall of The Franklin Institute, at the Stated Meeting held May 21, 1947.

This trend is dangerous to our national economy.

To give some background; in the four decades preceding the war the average yearly rate of increase in output per man hour in all manufacturing was somewhere between 3 to 3½ per cent. This permitted the substantial gain in real wages and in the standard of living of the American worker in this period.

This steady upward trend of productivity was halted and reversed by World War II. We have a long serious task to re-establish the prewar annual gain in productivity of 3 to 3½ per cent., much less to make up for the lost time during the war years. The way we do it is going to have a direct effect on the cost of goods we buy.

In the face of reduced output per man hour, what has happened to wages? There is no one in this audience unfamiliar with the answer. A few illustrations will show the extent of the increases. The settlement recently agreed to by the United States Steel Corporation raises their average hourly earning rate to about $1.47. In January 1941, straight time average earnings were 85¢. The total increase is 62¢ per hour or 73 per cent.

The average weekly earnings in all branches of industry have risen sharply since 1939. In January 1947 the average weekly earnings in soft coal mining were 280 per cent. of 1939, in the leather industry they were 250 per cent. of those in 1939; paper products approximately 200 per cent.; and in all manufacturing together the average weekly earnings had increased to about 170 per cent. of 1939 levels and these figures do not reflect any 1947 increases.

The significance of these fantastically large increases during a period in which output per man hour was stationary or declining lies in their impact on labor costs per unit of output. In the same five year period from 1939 through 1944, even before the record breaking two rounds of wage increases, unit labor costs in peacetime products went up 42 per cent.

The double round of wage increases unaccompanied by increases in output per man hour can only result in further increasing unit labor costs. Representing, as they do, such a very large element of total production costs, these increased unit labor costs make the prospects for any price reduction very bleak to say the least. The increases of the past two years are a payment in advance for an increase for six years at the prewar rate of production improvement.

The very best we could do in the most productive period in our history was an increase of about 6 per cent. per year in the 1919-1923 period. Unless labor is ready to take off its wraps and increase productivity to its maximum rate—to counterbalance postwar labor increases and to make up for the war period, we face a probability of a dropping standard of living.

The thought immediately comes to one's mind, how is this to be done? I think the answer is to get back to the fundamentals of industrial management. Collective bargaining procedures have developed to the point where the pay earned by a workman is specified to a high degree of accuracy with a multitude of clauses and governmental rulings to cover every contingency that may arise in his working day. The moment a workman climbs from a low bay crane to a high bay crane, there is a claim for a higher rate of pay.

The statement of worker responsibility as opposed to the rigid and legalistic wage schedule specifications governing employers is a very simple statement in the standard union agreement (including our own). A fair day's work for a fair day's pay—simple but what does it mean? Just what is a fair day's work and is industry getting it?

I said earlier that we must get back to the fundamentals of industrial management. The question we face today—"What is a fair day's work?" is the one for which Frederick W. Taylor sought an answer a number of years ago at the Midvale plant. Taylor said, "The principal object of management should be to secure the maximum prosperity for the employer coupled with the maximum prosperity for each employee." The success of the principles of industrial management developed by Taylor is due to his search for an answer to what constitutes a "fair day's work."

If Charles Day were alive today, I have no doubt that he would select this subject to be discussed in the address dedicated as a memorial to him. No person in his time exceeded his devotion to sound management, to the welfare of the employee, to the increase of productive output of our nation for the good of all. Scientific management was first developed and nurtured in the Philadelphia environment of which he was a leader.

We have, in my opinion, abondoned many of the principles of production control that were established by Taylor and those other pioneers Gantt and the two Gilbreths. I do not wish to imply that the art of scientific management is a lost art. It is still practiced with a high degree of effectiveness by the managements of a number of our Philadelphia plants. Leeds and Northrup, which has been properly recognized by various awards for achievement, comes immediately to mind. Atlantic Refining, Scott Paper, DuPont and others are leading practitioners of the art.

In general, however, the effectiveness of management has diminished very greatly from the peak of efficient production. So much so, that a thorough re-appraisal is justified. The heart of scientific management in the days of Taylor was the definition of a fair day's work with a system for giving a workman additional compensation for meeting or exceeding that "fair day's work." The "bogie" was based on time study and represented a quantity of output that could readily be done by a skilled workman in a day.

Even before World War I there were some crude incentive systems in operation. Some companies had a so called contract system such as Sam Vauclain operated at Baldwin. He would sell a group of locomotives and then the boss painter, boilermaker and others would bid for their part of the work as if they were independent contractors. Since these "contractors" in turn paid their men what they could get away with, it could hardly be said to be a scientific or fair system. The incentive was primarily for the contractor not for the workman.

The real factual fair time payment plans were developed during and immediately after the World War era when our country took the lead in world production. The old time memory and butcher book systems gave way to the more intelligent production systems that are the basis of present day factory management. Concurrently came the enthusiastic acceptance and installation of incentive plans based on time study data. Piece work, Gantt task and bonus, premium plans, standard time, all were directed to give a workman additional compensation for turning out a fair day's work. They were, for the most part, simple, direct and effective. The more a worker produced, the more he earned.

At the same time, a parallel development in production methods was taking place and one equally as revolutionary and equally destined to place America in the position of the most productive nation in the world. This was the production line developed by Henry Ford in 1913. The moving production lines became mechanical pace setters. A workman had to perform certain operations in the time it took for the line to move past him. The task was still, therefore, a given quantity of work to be performed in a given period of time. While earnings were not tied directly to output as in incentive plans, you will recall that Ford paid the highest day rate in history.

The production line completely revolutionized industries where there is a mass production of products such as automobiles, refrigerators, radio sets, electric switches, etc.

The two systems of incentive work and production line combined in the early twenties to produce the most rapid productivity increase this country has ever experienced. In the period from 1919 to 1939 the increase in output per man hour brought about a 40 per cent. drop in unit labor costs thus permitting an average increase in hourly earnings of over 25 per cent. and reduced prices to consumers.

These figures demonstrate the success of these various plans for production control based on the principle of a fair day's work for a fair day's pay. The improvement and refinement in production methods continued until the late 1930's.

After 1936, which is considered by many to be our most efficient year, the high efficiency of incentive plans commenced to slip. It is not mere chance that the period of this decline coincides with the large expansion of the labor movement in the United States.

Several factors have led to a deterioration of incentive plans and productive effort. The war with its tremendous demand for manpower brought inexperienced and inefficient workers into industrial plants. Normal output standards could not be applied to such workers. Some companies made the serious error of easing their standards to artificially give higher earnings in order to attract additional workers. Plant wide incentive plans were introduced in companies like Grumman Aircraft, Electric Boat Company and others in late 1943. These plant wide plans had the failing that the individual worker was far removed from the "task" that was set. He could not see a direct relationship between his output and his reward.

Many companies weakened the effect of their incentives by including indirect workers such as crane men, storekeepers, sweepers, truckers, inspectors and the like. Frequently this inclusion resulted in a flat percentage or bonus added to the weekly earnings—a percentage which most such companies now discover is a fixed part of their wage structures.

In the past few years flat increases have been granted to be added to piece work earnings of workmen. These have amounted to a total of as much as 50¢ an hour in some plants and form a substantial part of the workman's pay. The guaranteed day rate plus this 50¢ bonus are so large as to overshadow the amount proportioned to incentive effort. The urge to produce is not as great as before.

The greatest step in the weakening of incentive systems was when they were brought into collective bargaining. The scientific approach to determining a fair day's work begun by Taylor was replaced by an approach based on compromise and determined by a trial of strength at the bargaining table. A labor union, by its very nature, is not in a position to take the responsibility of setting proper standards. Unless a union leader is a fool—and few of them are—he will not risk grievances from his membership with respect to standards of output where he participated in their determination. When the union is granted an equal voice in rate setting, the union representative must play it safe for his own protection.

The standards upon which incentive systems rest must be determined objectively and not from a trial of relative bargaining strength. The suspicion with which many workmen already view company set standards can only be confirmed if—without any justification other than union pressure—standards are changed through bargaining. Standards should be set objectively and challenged on the basis of objectivity alone.

Although there are many plants where incentive systems are still fairly effective, a fair appraisal of the whole field of incentive application indicates that such systems have lost much of their original effectiveness.

The question may be asked how to abandon or improve one of these wage incentive plans that has gone astray and gotten out of hand. The only road I know is to mutually agree to place the plan back to a sound basis using scientific methods procedures of the past. Unless this is done, both the plant and the worker are going to get hurt.

There is an equally important group in industry who are now paying fixed hourly rates without any incentive plan or a measured work system. The question these companies are asking is, "How are we to obtain a fair day's work for the high hourly rates we pay?" Without some standard to serve as a work measurement, the output of any worker tends to fall to the level of the poorest of the group. A good workman soon wonders why he should produce more work in a day than a poor, lazy or indifferent workman who receives the same rate. Since, under the single day rate system, his earnings are not adjusted to his output, the natural reaction is to adjust his output to that of the less efficient worker. If he has any desire to produce at a higher rate, he is soon discouraged by his associates.

I believe that the answer to this problem is to set up standards of performance and adjust the hourly rates of individuals up or down in accordance with their output; in other words measured day work.

A sound measured day work plan has three prerequisites. The first is a careful program of job evaluation by which the relative values of the various jobs are determined by comparing the skill, responsibility, effort and working conditions inherent in them.

The application of job evaluation plans has been widely extended in the last decade. The war gave impetus to this extension largely through the influence of the War Labor Board. The steel industry, for example, is today completing a comprehensive job evaluation ordered by a Board directive back in 1943. The plans adopted in this industry have been mutually accepted by both management and labor. Both parties are equally convinced that job evaluation is an absolute essential for sound working relationships.

The second prerequisite for a measured day work plan is the careful establishment of measures of output for each operation. Again, you see the definition of a fair day's work. These must be established by scientific methods and not by mutual consent nor by collective bargaining.

The third prerequisite for such a plan is a provision for an employee's rate to be adjusted up or down according to his output. For instance, the drill press operator who turns out the standard number of pieces over a period of time will receive the standard rate established by job evaluation; the operator who regularly produces an amount exceeding the standard by a certain percentage will be paid a premium rate; the operator who falls short of the standard will receive a rate lower than standard; and the operator who regularly fails to produce a certain minimum output will be removed.

There are a few plants, particularly in heavy industry, where the work is not sufficiently repetitive to justify any premium plan. This is particularly true in the case of a large shaft which may require two weeks to machine. In such plants it will be necessary to set up standards of allowed hours based on past experience and implement those standards with first class management supervision. Mechanics who do not meet those standards are either dismissed or derated to a lower classification.

While this discussion has been frank, it is not intended to be controversial, but merely to present the present day production situation. It is in the interest of worker, management and the nation as a whole alike that output be increased.

To sum up, output per man hour is not increasing. Even before the two rounds of general increases, labor costs per unit were rising rapidly. The increases of the past two years cannot be offset by increased productivity for some years to come, even if we again achieve outputs at the highest rates obtained in preceding decades. Since the only source of higher real wages is increased output, we must bend every effort to the end that output per man hour increases.

To do this we must get back to the fundamental principle of "a fair day's pay for a fair day's work." For various reasons, incentive plans—so effective in prior years—have been weakened and distorted. Single hourly rates without any measure of productivity do not assure a fair day's work in exchange for the wages paid.

Where incentive systems are still in use, they must be brought back to a sound basis.

Where day rates are the rule, they should be established by sound job evaluation, coupled with standards of output. The workman's performance is to be measured against that standard.

Such a program is the main hope for the next decade. A fair day's work is again defined and not as a meaningless generality. One of the programs outlined can be applied to any type of industrial plant. Production and productivity will increase to serve as a counter move against the inflationary push of rising unit labor costs.

America has reached a point where everyone can be provided with a decent living. We have more than enough industrial and agricultural capacity to support a large increase in total production. The only limiting factor is the willingness of our labor force to produce. If we can meet the challenge of productivity, we can move on toward constantly rising living standards for everyone. Never before has a nation been so close to abolishing poverty and to meeting all the material demands of its citizens.

The leaders of labor must support such a program of increased production unless they deliberately ignore the effects of inflationary wage rates upon the workman's standard of living.

I am sufficiently an optimist to predict that the increasing trend of wages in the face of stationary or declining productivity has reached its peak and that the next decade will show a definite increase in the output per man hour. Management must obtain greater production. Labor must cooperate to this end. The result will be to translate into real wages the increases obtained in the past two years.

There are already indications that our great factories will be more productive in this year of 1947 than in any period since 1940.

NUMERICAL TRANSFORMATION PROCEDURES IN CONTINUOUS BEAM ANALYSIS.

BY

STANLEY U. BENSCOTER

SYNOPSIS.

Several new types of numerical procedures are presented for the calculation of bending moments in continuous beams. Basic physical properties of the beam and its loading, which are familiar to the engineer, are employed in the procedures. These basic properties are defined and may be calculated conveniently by means of a conjugate beam. The procedures bear some resemblance to the process of moment distribution. A mathematical demonstration of the validity of the procedures is given in the Appendix.

INTRODUCTION.

The analysis of a continuous beam may be performed by solving the equations of three slopes and substituting this solution into the slope-deflection equations. When this method of solution is resolved into a numerical process of successive corrections the natural result is moment distribution as introduced by Hardy Cross.[1] Many other approximation methods of calculation have been proposed but have not been generally adopted.

In contrast, the analysis of a continuous beam may be performed by solving the well-known equations of three moments. Since this method involves the solution of only one system of equations, many methods of solution are available for use after the equations have been written down. It is the object of this paper to present several numerical approximation processes, based on natural physical constants, which can be easily learned and performed without reference to the equations which govern the solution.

NOTATION.

L = span length,
m = bending moment,
m_s = statically determinate moment,
m_i = indeterminate moment,
t = transformation factor,
δ = angular dislocation,
θ = end-slope of beam,
ϕ = primary flexibility,
γ = secondary flexibility.

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[1] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by H. Cross, *Trans. A.S.C.E.*, Vol. 96, 1932, pp. 1–156.

PHYSICAL PROPERTIES OF SINGLE SPAN.

The analysis of a continuous beam must begin with the determination of physical constants related to a single span. These physical constants arise as coefficients and constant terms in the equations of three moments. Referring to Fig. 1(a), the left-end flexibility may be defined as the angle of rotation of the beam at the left end caused by a unit moment applied at that end, when the beam is simply supported. The left-end flexibility is indicated as $-\phi_a$ in Fig. 1(a). In a similar manner the right-end flexibility may be defined and is shown as ϕ_b in Fig. 1(b). Counterclockwise rotation is regarded as positive in this paper. The angular rotation of the beam at the opposite end from that at which the unit moment is applied may be called the secondary flexibility. This angle is indicated as γ in Fig. 1(a) and as $-\gamma$ in Fig. 1(b). The same magnitude arises for the secondary flexibility in both cases in agreement with Maxwell's law of reciprocal deflections. These flexibilities may be calculated conveniently by using a conjugate beam.[2]

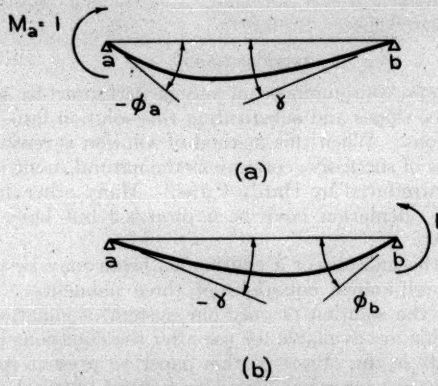

FIG. 1. Definition of flexibilities.

The flexibilities are solely dependent upon the physical properties of the beam. Two additional physical constants of the single span must be determined. These are the two end-slopes of the beam which would occur, under the action of the applied load, if the span were acting as a simple beam. These two constants are dependent both on the applied load and the physical properties of the span. These end-slopes are indicated as θ_a and θ_b in Fig. 2. They may also be calculated by using the conjugate beam.

---

[2] "Deflection of Beams by the Conjugate Beam Method," by H. M. Westergaard, J. Western Soc. Engrs., Dec. 1921.

ANGULAR DISLOCATION.

Consider a two span beam as shown in Fig. 3(a). The first step in the analysis consists of calculating the flexibilities and simple beam end-slopes for each span. This needs to be done only at the center support. In general, these quantities do not have to be computed at external supports of a continuous beam. The beam is considered to be cut at the center support, and to form two simple beams as shown in Fig. 3(b). The joints, or supports, may be numbered in any convenient manner, such as shown in Fig. 3(a). The end-slopes, and other physical constants, may be indicated by using double subscripts such that the first number indicates the support and the second number, combined with the first, indicates the span.

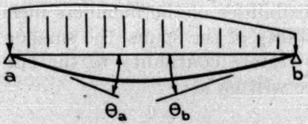

FIG. 2. Simple beam end slopes.

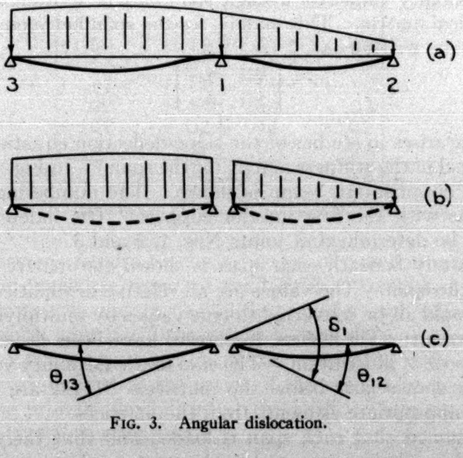

FIG. 3. Angular dislocation.

Because of the cutting of the beam at the center support there arises an angular dislocation of the elastic curve which is indicated as δ_1 in Fig. 3(c). This angular dislocation is a direct violation of the law of continuity. The internal joint moment which must exist, in order that continuity shall be preserved, is directly proportional to this angular dislocation. The magnitude of δ_1 is obtained as the algebraic difference of the end rotations θ_{12} and θ_{13},

$$\delta_1 = \theta_{13} - \theta_{12}. \tag{1}$$

SOLUTION OF TWO SPAN BEAM.

The bending moment at the center support is inversely proportional to the sum of the flexibilities of the members meeting at the support. The greater the flexibility of the spans, the smaller will be the internal moment required to restore continuity to the cut beam. Hence the joint moment may be written as

$$m_1 = \frac{-\delta_1}{\Sigma_1\phi}. \tag{2}$$

The subscript in each factor indicates the number of the joint. The quantity $\Sigma_1\phi$ is the sum of the flexibilities of the members meeting at the joint. This equation may be recognized as the equation of three moments for a two-span beam on simple supports.

MOMENT TRANSFORMATION BY SUCCESSIVE CORRECTIONS.

The flexibility values for a single span may be written in the form of a symmetrical matrix. This matrix, for the span between joints Nos. 1 and 2, may be written as

$$\begin{bmatrix} \phi_{12} & \gamma_{12} \\ \gamma_{21} & \phi_{21} \end{bmatrix}.$$

This matrix arises in studies of the slope-deflection equations since it is the reciprocal of the stiffness matrix for the span.[3] In Fig. 4 a four-span continuous nonprismatic beam is shown. The numbering of the supports begins with the first interior support. The indeterminate moment must be determined at joints Nos. 1, 2 and 3.

Immediately beneath each span is shown the matrix of flexibility values for the span. The values are all relative flexibilities, being such that they could all be converted to true values by multiplying by a conversion constant. The values, as shown, have been chosen arbitrarily for the purpose of illustration. The sum of the flexibility values at each support are shown just below the matrices. These are obtained by adding the appropriate elements from the matrices.

It is assumed that each span is loaded and that the simple beam end-slopes have been computed to have the values at each interior support as shown. The difference between these values at each support gives the angular dislocation. These values are then divided by the sum of the flexibilities at each joint to obtain an initial estimate of the joint moment.

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[3] "Matrix Analysis of Continuous Beams," by S. U. Benscoter, *Proc. A.S.C.E.*, Vol. 72, no. 8, Oct. 1946, pp. 1091–1122.

$$m' = \frac{-\delta}{\Sigma\phi}. \tag{3}$$

The initial estimate m' is computed in the same manner as has been previously demonstrated for the center joint moment of a two-span beam. Thus the analysis of a two-span beam provides a fundamental process which must be learned before proceeding to the analysis of beams of more than two spans. The numerical calculation procedure to be explained transforms the initial estimates, m', into the true joint moments, m.

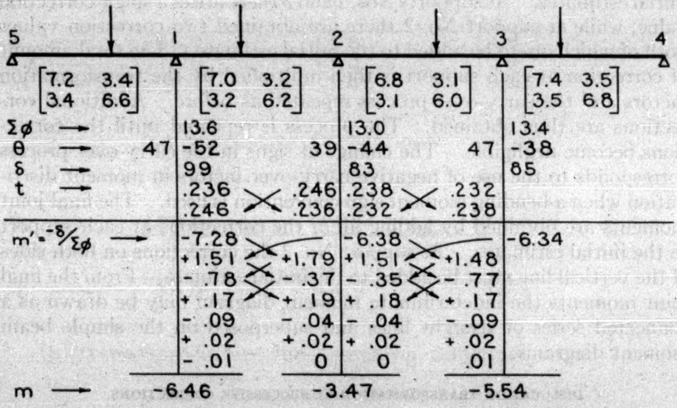

5
1
2
3
4
7.2 3.4
3.4 6.6
7.0 3.2
3.2 6.2
6.8 3.1
3.1 6.0
7.4 3.5
3.5 6.8
Σφ →
13.6
13.0
13.4
θ →
47
-52
39
-44
47
-38
δ →
99
83
85
t →
.236
.246
.246
.236
.238
.232
.232
.238
m′=-δ/Σφ →
-7.28
+1.51
- .78
+ .17
- .09
+ .02
- .01
+1.79
- .37
+ .19
- .04
+ .02
0
-6.38
+1.51
- .35
+ .18
- .04
+ .02
0
+1.48
- .77
+ .17
- .09
+ .02
.01
-6.34
m →
-6.46
-3.47
-5.54

FIG. 4. Transformation of moments by successive corrections.

The calculations require the introduction of transformation factors which may be defined in terms of flexibilities. Associated with each joint there are two secondary flexibilities, one for each adjacent span. For example, with joint No. 2 the values γ_{21} and γ_{23} may be associated. There are also two transformation factors associated with each joint. These factors are defined as the ratios of the secondary flexibilities to the sum of the primary flexibilities at the joint. Thus, at the second joint, if t is a transformation factor,

$$t_{21} = \frac{\gamma_{21}}{\Sigma_2\phi}, \tag{4a}$$

$$t_{23} = \frac{\gamma_{23}}{\Sigma_2\phi}. \tag{4b}$$

Using the flexibility values from the matrices, the transformation factors have been computed to be as shown in Fig. 4. Before the calculations are begun, the transformation factors must be transposed in each span as indicated by the crossing arrows.

A horizontal line is drawn beneath the transformation factors and the initial estimates are written beneath this line. The transformation calculation is performed in the same manner as the carry-over step of a moment distribution cycle.

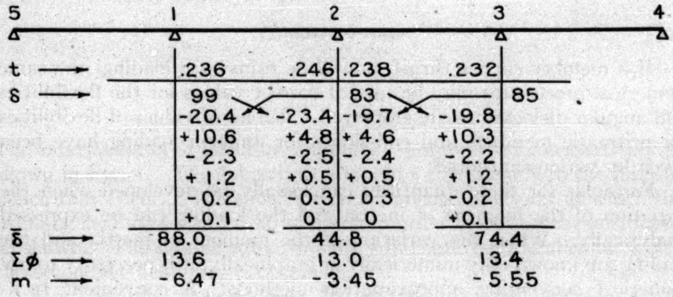
Each initial estimate is multiplied by the transformation factors just above it. The resulting quantities, with signs changed, are carried across the span to the opposite end of the member. The new quantities thus established at each support are corrections to be added to the initial estimates. At supports Nos. 1 and 3 there arises a single correction value, while at support No. 2 there are obtained two correction values both of which are to be added to the initial estimate. The total amount of correction at each support is then multiplied by the transformation factors and the carry-over process repeated as before. Additional corrections are thus obtained. The process is repeated until the corrections become negligible. The change of signs in the carry-over process corresponds to the use of negative carry-over factors in moment distribution when a bending moment sign convention is used. The final joint moments are obtained by adding all of the corrections at each support to the initial estimates. At support No. 2 the corrections on both sides of the vertical line must be added to the initial estimate. From the final joint moments the indeterminate moment diagram may be drawn as a connected series of straight lines and superposed on the simple beam moment diagrams.

DISLOCATION TRANSFORMATION BY SUCCESSIVE CORRECTIONS.

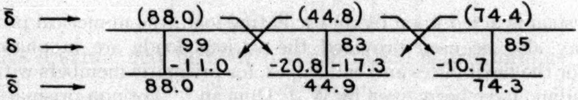
The calculations may be performed in a somewhat different manner as illustrated in Fig. 5(a). In this process the angular dislocations are transformed into new values indicated as $\bar{\delta}$. The transformed dislocation at each support is divided by the sum of the flexibilities at the support to obtain the final joint moment.

The beam of Fig. 5 is the same as that of Fig. 4 and the values of the transformation factors and dislocations have been copied. In the process of Fig. 5 the transformation factors are not transposed. The convenience of not having to remember to transpose the transformation factors causes the process of dislocation transformation to appear preferable.

The cycles of carry-over computations are performed in the same manner as in Fig. 4. The transformed dislocation is obtained at each support by adding all of the corrections to the true value. The transformed dislocation at each support is then divided by the sum of the flexibilities at the support to obtain the joint moment. This procedure is similar, from an algebraic viewpoint, to the relaxation method developed by R. V. Southwell.[4]


5
1
2
3
4
t →
.236
.246
.238
.232
δ →
99
-20.4
+10.6
- 2.3
+ 1.2
- 0.2
+ 0.1
-23.4
+4.8
-2.5
+0.5
-0.3
0
83
-19.7
+ 4.6
- 2.4
+ 0.5
- 0.3
0
-19.8
+10.3
- 2.2
+ 1.2
-0.2
+0.1
85
δ̄ →
88.0
44.8
74.4
Σφ →
13.6
13.0
13.4
m →
-6.47
-3.45
-5.55

(a) Successive Corrections


δ̄ →
(88.0)
(44.8)
(74.4)
δ →
99
-11.0
-20.8
83
-17.3
-10.7
85
δ̄ →
88.0
44.9
74.3

(b) Iteration Cycle for Checking Solution

FIG. 5. Transformation of angular dislocations.

TRANSFORMATION BY ITERATION.

When one, and only one, system of linear algebraic equations is being solved, a numerical transformation process can be performed either by calculating successive corrections or by iteration. An iteration solution of a continuous beam may be obtained by transforming moments or by transforming angular dislocations. An example of numerical transformation by iteration has been given elsewhere[5] in connection with the calculation of shear flows in multicellular beams. The same calculation procedure may be followed in the analysis of continuous beams but will not be illustrated. However, a single cycle of iteration is shown in Fig. 5(b). This cycle serves merely to check the preceding calculations. A process of successive corrections must be checked in some manner. This combination method in which a solution by successive corrections is followed by a cycle of iteration is a rapid practical method for design purposes.

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[4] "Relaxation Methods in Engineering Science," by R. V. Southwell, Oxford Univ. Press, 1940.

[5] "Numerical Transformation Procedures for Shear Flow Calculation," by S. U. Benscoter, *J. of Aero. Sci.*, Vol. 13, no. 8, Aug. 1946, pp. 438–443.

BEAM-COLUMNS.

If a member carries thrust as well as transverse loading, the same numerical procedures may be used if correct values for the flexibilities and angular dislocations are employed. Tables of values of flexibilities for prismatic members and end-slopes for uniform loading have been given by various authors.[6]

Formulas for these quantities can usually be developed when the variation of the moment of inertia and the loading can be expressed analytically. When the variation of the moment of inertia and the loading are known only numerically or graphically, it is necessary to use numerical converging approximation methods. A convenient technique for writing such calculations has been given by N. M. Newmark.[7] The question of critical buckling load is considered in the Appendix.

VIBRATORY LOADING.

If each span is acted upon by an oscillating load, the numerical procedures may also be used provided the various loads are in phase. Formulas for the flexibilities and end-slopes, for prismatic members with uniform loading, have been given by W. J. Duncan.[8] For non-prismatic members, with arbitrary loading distribution, numerical methods would again be required to determine the physical constants. The question of resonant frequency is also considered in the Appendix.

CONCLUSIONS.

Several methods of performing the calculations for the solution of continuous beams have been described. These processes correspond to various algebraic techniques which might be employed in solving the equations of three moments. The methods may be referred to as numerical transformation procedures. Such procedures become especially useful for design purposes when the various factors which enter into the calculation can be shown to have a simple physical significance as in the present case.

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[6] See, e.g., "Airplane Structures," by A. S. Niles and J. S. Newell, Vol. II, John Wiley & Sons, Inc., New York, p. 72.

[7] "Numerical Procedure for Computing Deflections, Moments and Buckling Loads," by N. M. Newmark, *Trans. A.S.C.E.*, Vol. 108, 1943, pp. 1161–1234.

[8] "Free and Forced Oscillations of Continuous Beams; Treatment by the Admittance Method," by W. J. Duncan, *Philos. Mag.*, Ser. 7, Vol. 34, no. 228, Jan. 1943, pp. 49–63.

APPENDIX.

MATHEMATICAL DEVELOPMENT.

Considering a four-span beam on five simple supports, the equations of three moments appear as follows:

$$(\Sigma_1\phi)m_1 + \gamma_{12}m_2 = -\delta_1, \tag{5a}$$

$$\gamma_{21}m_1 + (\Sigma_2\phi)m_2 + \gamma_{23}m_3 = -\delta_2, \tag{5b}$$

$$\gamma_{32}m_2 + (\Sigma_3\phi)m_3 = -\delta_3. \tag{5c}$$

The intermediate, or effective, supports are numbered consecutively as shown in Fig. 4. The subscript on m and δ corresponds to the joint at which they occur. The subscript on Σ indicates the joint at which the primary flexibilities are to be summed in order to obtain the coefficient as shown. The double subscripts on γ indicate the member for which the secondary flexibility is obtained.

After dividing through Eqs. (5) by the diagonal coefficients, the resulting equations may be written in the following expanded matrix form:

$$\begin{bmatrix} 1 & t_{12} & 0 \\ t_{21} & 1 & t_{23} \\ 0 & t_{32} & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} m_1' \\ m_2' \\ m_3' \end{bmatrix}. \tag{6}$$

This equation may be written in contracted matrix form as

$$\{[I] - [T]\}[m] = [m'], \tag{7}$$

where

$$[T] = -\begin{bmatrix} 0 & t_{12} & 0 \\ t_{21} & 0 & t_{23} \\ 0 & t_{32} & 0 \end{bmatrix}. \tag{8}$$

Multiplying through Eq. (7) by the reciprocal matrix $\{[I] - [T]\}^{-1}$ gives the solution in the form

$$[m] = \{[I] - [T]\}^{-1}[m']. \tag{9}$$

Just as in the case of scalar numbers, the reciprocal of a matrix binomial may be expressed in terms of a power series. Hence, Eq. (9) may be written as

$$[m] = \{[I] + [T] + [T]^2 + [T]^3 + \cdots\}[m'] \tag{10}$$

or

$$[m] = [m'] + [T][m'] + [T]^2[m'] + \cdots. \tag{11}$$

The second and additional terms of this series are column vectors of successive corrections to be added to the vector, $[m']$, of initial estimates. It may be shown [5] that the successive terms of this series correspond to the successive cycles of computations illustrated in Fig. 4. Hence, the calculation procedure yields a correct solution.

A development of the validity of an iteration procedure may begin with Eq. (7). Removing the brace gives

$$[m] - [T][m] = [m']. \tag{12}$$

Transposing the second term gives

$$[m] = [m'] + [T][m]. \tag{13}$$

The second term on the right hand side is a vector of corrections to be added to the initial estimates in order to obtain an improved estimate. Using the initial estimates as an assumed solution and substituting on the right hand side gives the improved estimates $[m'']$.

$$[m''] = [m'] + [T][m']. \tag{14}$$

If this improved estimate is substituted into Eq. (13) the result is

$$[m'''] = [m'] + [T][m''] \tag{15}$$

$$= [m'] + [T][m'] + [T]^2[m']. \tag{16}$$

By continuing this iteration process indefinitely a series of terms may be formed. Comparing Eq. (16) with Eq. (11) indicates that the series thus formed will be the same as that previously obtained. Thus the iteration process leads to the correct solution.

The question of convergence of the power series of matrices in Eq. (10) must be considered. It is always found, when particular cases are examined, that the series converges for beams with transverse loading only. In the case of beam–columns the series will fail to converge if the thrust has a critical, or characteristic, value. Similarly, with vibratory loading the series will fail to converge if the frequency of the applied load is of a resonant, or characteristic, value. Thus it is seen that a criterion for determining whether or not the series will converge is desirable.

A matrix $[T]$ is said to be similar [9] to a matrix $[N]$ if a matrix $[P]$ can be found which satisfies the equation

$$[T] = [P][N][P]^{-1}. \tag{17}$$

Similar matrices have the same latent roots and the same determinant. A square non-singular matrix $[T]$ is similar to one and only one diagonal matrix. This diagonal matrix contains the latent roots of $[T]$ and, hence, the matrix $[N]$ may be assumed to be a diagonal matrix containing the latent roots of $[T]$. In this case the matrix $[P]$ contains the latent column vectors [10] of $[T]$ while the matrix $[P]^{-1}$ contains the latent row vectors of $[T]$.

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[9] "Vectors and Matrices," by C. C. MacDuffee, Math. Assn. of Am., 1943, p. 112.

[10] "Determinants and Matrices," by A. C. Aitken, Interscience Pub., Inc., N. Y., p. 89.

The powers of $[T]$ are needed for substitution in the power series

$$\begin{aligned}[T]^2 &= [P][N][P]^{-1}[P][N][P]^{-1} \\ &= [P][N]^2[P]^{-1}, \qquad (18)\end{aligned}$$

$$[T]^3 = [P][N]^3[P]^{-1}, \qquad (19)$$

etc.

Substitution in Eq. (10) gives

$$[m] = [P]\{[I] + [N] + [N]^2 + \cdots\}[P]^{-1}[m']. \qquad (20)$$

The convergence of the original series in $[T]$ now depends upon the convergence of a series of diagonal matrices. The power of a diagonal matrix is computed by raising each individual diagonal element to that power. Thus each matrix within the brace is a diagonal matrix and the series is equal to a diagonal matrix. The sum of the series may be indicated as $[\Lambda]$,

$$[\Lambda] = [I] + [N] + [N]^2 + [N]^3 + \cdots. \qquad (21)$$

If λ_1 is the largest element of $[\Lambda]$ and n_1 is the largest element of $[N]$, then

$$\lambda_1 = 1 + n_1 + n_1^2 + n_1^3 + \cdots \qquad (22)$$

$$= \frac{1}{1 - n_1}. \qquad (23)$$

From this formula it is seen that when $n_1 = 1$ the power series of matrices will fail to converge to finite values. Thus the criterion for convergence can be stated. The numerical transformation procedures will converge to give a finite solution if the largest latent root of the transformation matrix $[T]$ is less than unity. The attainment of unity by the largest latent root of $[T]$ corresponds to the vanishing of the determinant of the original equations. Both conditions indicate the existence of characteristic values of buckling or vibratory loads. A similar criterion can be established for the convergence of moment distribution. Studies along this line have been made by R. Oldenburger.[11]

The validity of the processes of moment transformation has been demonstrated. The validity of the processes of angular dislocation transformation may be demonstrated by similar arguments. If the transformed dislocations are known, the bending moments may be computed from the formula for a two-span beam,

$$m_i = \frac{-\bar{\delta}_i}{\Sigma_i \phi}. \qquad (24)$$

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[11] "Convergence of Hardy Cross's Balancing Process," by R. Oldenburger, *J. Appl. Mech.*, Vol. 7, no. 4, pp. A-166 to A-170.

If this formula is substituted into Eqs. (5), they may be written in the following expanded matrix form,

$$\begin{bmatrix} 1 & t_{21} & 0 \\ t_{12} & 1 & t_{32} \\ 0 & t_{23} & 1 \end{bmatrix} \begin{bmatrix} \bar{\delta}_1 \\ \bar{\delta}_2 \\ \bar{\delta}_3 \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}. \tag{25}$$

This equation may be written in contracted matrix form as

$$\{[I] - [T]'\}[\bar{\delta}] = [\delta]. \tag{26}$$

The matrix $[T]'$ is the transpose of $[T]$. As in the previous development the reciprocal of the binomial within the brace may be expressed as a power series,

$$[\bar{\delta}] = \{[I] + [T]' + [T]'^2 + [T]'^3 + \cdots\}[\delta]. \tag{27}$$

According to this formula the transformed dislocations may be obtained by adding to the original dislocation values a series of corrections given by the terms of the series. The successive terms of this series may be shown [5] to correspond to the successive cycles of computations in Fig. 5. Hence, this calculation process must also give a correct solution.

It is apparent that the process of iteration may be developed for dislocation transformation in the same manner as presented for the transformation of moments. The convergence of the power series of Eq. (27) depends upon the same criterion as before since the latent roots of a matrix and its transpose are identical.

A PRINCIPAL STRESS METHOD OF STRESS ANALYSIS.

BY

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SYNOPSIS.

The principal stress differences $(P - Q)$ and $(P - R)$ of the Lamé equations and their constant value lines are identified by the new words *detend*, *double-detend* and *equidetend*. Principal stresses are defined as real components of stress. The $\partial Xx/\partial x$ terms of the Cauchy equations with all derived terms such as shear are shown to be imaginary components of force per unit area.

A *gradient triangle* whose sides ΔP, ΔQ, and $\Delta(P - Q)$ represent the maximum gradients of P, Q, and $(P - Q)$, is developed graphically. Five equations defining elements of the gradient triangle in terms of five measurable quantities $(P - Q)$, $(P - Q)/\Delta(P - Q)$, $ds_3/d\alpha$, ω and θ, are given. The normal distance ds_3 between two near isoclinics, α and $\alpha + d\alpha$, is divided by $d\alpha$. The angles ω and θ are respectively measured between a stress axis line taken in the direction of increasing stress and an isoclinic and $\Delta(P - Q)$.

ABSTRACT.

The equations of equilibrium in a stress field as given by Lamé are compared with those given by Cauchy. The $(P - Q$ and $P - R)$ terms of the Lamé equations identified by the new words *detend* and *double detend* are invariant for any point in a stress field. The principal stresses are defined as real stress components. The $\partial Xx/\partial x$ terms of the Cauchy equations are considered as imaginary force components. Real and imaginary stress and force components are differentiated by the differences between ray vectors, as for example a force which has only one real value and direction, and diffuse vectors, as for example a gradient, which has a real value in any direction. Shear is differentiated from *detend* value by its variation with the direction in which it is taken. Shear is found to be an imaginary component of force parallel to a plane.

The photoelastic *equidetend* (isochromatic) and isoclinic maps of a plane stress field define, for any point, a *gradient triangle* with the three maximum gradients, $\Delta(P - Q)$ ΔP and $\Delta - Q$, for its sides. Five quantities, the *detend* $(P - Q)$, the ratio of the *detend* to its gradient (r_4), the ratio (r_3) of the isoclinic normal ds_3 to the stress axis direction change along its length $d\alpha$, the direction of the *equidetend* normal (θ) and the orientation of an isoclinic line (ω) with reference to a principal stress direction, determine each *gradient triangle*. Five equations which define elements of the *gradient triangle* in terms of the five measurable quantities, are developed. Definition of the triangle can be complete with two of the five measurable quantities or two of the five equations omitted. Graphic integration of principal stress changes along any line in a stress field can be made by plotting the line straight and measuring the area swept out by the normal projections, along the line, of the sides of the gradient triangle, after it has been rotated through 90 degrees.

Properties of the gradient triangle are used to solve several symmetrical stress problems and to integrate along two lines through a stress field.

INTRODUCTION.

Photoelastic tests of a plane stress field in a transparent elastic test member give, by direct observation, what are called isochromatic and isoclinic maps. These are invariant with respect to the scale dimensions

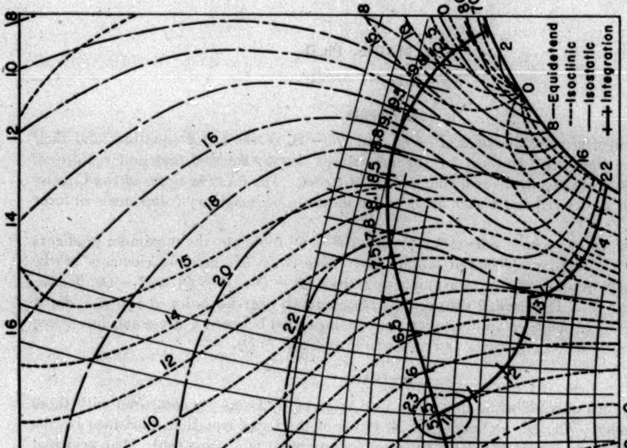
8—Equidetend
Isoclinic
Isostatic
Integration
18
10
12
14
16
0
2
4
6
8
10
12
14
15
16
18
20
22
30
88
86
87
19
21

FIG. 1. Photoelastic data map of a section of a pyralin strip with a symmetrical circular opening. The strip was stressed by uniform extension of the ends in the direction indicated by the arrow. The equidetend (isochromatic) lines are valued by an arbitrary scale. The isoclinic values are measured clockwise from the direction of the applied load. The line of integration does not start at the edge of the strip but it does end at the edge of the circular opening.

of the model, the elastic properties of the model material, the magnitude of the load applied, and hydrostatic pressure. These two maps combined give, for any point in the member, the data required to determine the stress gradients in any direction from that point.

Isochromatic Map.

When a transparent stress member is viewed in a field of circularly polarized light through the necessary analyzing unit, a colored or chromatic field is observed. Each tint of color represents a particular value of the *principal-stress difference* $(P - Q)$. Lines of uniform tint have been called isochromatic lines, and when isochromatic lines separated by fixed color-difference intervals are drawn, an isochromatic map is obtained. A map of this type is shown by the dot-dash lines of Fig. 1 which represents a photoelastic analysis of a stress field near a circular opening in a plate under uniform tension in the direction indicated by the arrow.

Isoclinic Map.

When a stressed field is viewed in plane polarized light through a crossed polarizer, the field appears dark wherever one principal stress has the orientation of the plane of polarization of the light. Dark lines seen in plane polarized light are called isoclinics. With rotation of the plane of polarization, the lines shift in position. When isoclinic lines are drawn to represent equal differences in the orientation of the plane of polarization, an isoclinic map is obtained. A map of this type is shown by the dotted lines of Fig. 1.

The isochromatic and isoclinic maps of a stress field include all of the observational data resulting from a photoelastic study and all of the data required for a complete analysis of all stress gradients in the field. If, in addition to these two maps, one principal stress or the sum of both principal stresses is known for one point in the field, a determination of all stress values for all parts of the field is possible. The additional required value serves to fix the hydrostatic pressure. The principal stresses P and Q vary with it but the isoclinic and isochromatic maps are invariant with changes in hydrostatic pressure; therefore, with reference to it, the maps may be called invariant whereas the principal stresses are variable.

Stress-Axis Map.

If two orthogonal sets of lines are so drawn that at any point in the field one of the two lines passing through the point has the orientation of one principal stress as shown by the isoclinic map for the corresponding point, a stress-axis map of the field is obtained. Such a map is shown by the orthogonal solid lines of Figs. 1 and 4. At every point of the plane field the stress-axis lines have the orientation of the two principal stresses P and Q for the corresponding point in the stress field.

In a three-dimensional stress field there are three systems of stress-axis lines corresponding to the three principal stresses P, Q, and R. These lines define three systems of stress-axis surfaces which are normal to the stress-axis lines. All force action across a stress-axis surface is normal to the surface. Three stress-axis planes at any point are tangent to the three stress-axis surfaces. The principal-stress differences and the radii of curvature of the projections of the stress-axis lines on the tangent planes constitute the geometrical elements which are used in the principal-stress method of stress analysis.

STRESS-COMPONENT ANALYSIS OF STRESS EQUILIBRIUM.

Stress-Component Equations.

Lamé [1]* in 1859 defined the conditions of equilibrium represented by the three maps of the preceding paragraph by the following rule:

> "In every system which represents stress equilibrium each of three principal elastic stresses undergoes a variation in its direction, which is equal to the sum of its differences with respect to the other two, multiplied respectively by the curvatures of the surface to which it applies."

Stated in the form of an equation Lamé's rule becomes:

$$\begin{aligned} \frac{dP}{ds_1} &= \frac{(P-Q)}{r_{12}} - \frac{(P-R)}{r_{13}}, \\ \frac{dQ}{ds_2} &= \frac{(Q-P)}{r_{21}} - \frac{(Q-R)}{r_{23}}, \\ \frac{dR}{ds_3} &= \frac{(R-P)}{r_{31}} - \frac{(R-Q)}{r_{32}}. \end{aligned} \tag{1}$$

P, Q, and R, see Fig. 3, are the orthogonal principal stresses, hereafter called simply stresses. Each r represents the radius of curvature of the projection of a stress-axis line on a stress-axis plane. The subscripts 1, 2, and 3 represent respectively the P, Q, and R orientation. The two subscripts of an r identify the stress-axis plane in which a projection lies. The second subscript identifies also the stress-axis line projected. For example, r_{12} is the radius of curvature of the Q stress-axis line projected onto the P, Q axis plane while r_{32} is the curvature of the projection of the same line on the Q, R plane.

An r is positive if the differentiation in the equation containing it is toward the center of curvature. This convention follows Lamé rather than the more recent practice of making r positive for differentiation away from the center.

Equation (1) for a plane stress field is simplified to the expressions commonly used in photoelastic analysis, namely:

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* Numbers in brackets indicate references on page 62 of this report.

$$\begin{aligned} \frac{dP}{ds_1} &= \frac{P - Q}{r_2}, \\ \frac{dQ}{ds_2} &= \frac{Q - P}{r_1}, \end{aligned} \tag{2}$$

where r_1 is the radius of curvature of the P stress-axis line and r_2 is that of the Q stress-axis line.

Equations (1) and (2) are the equations of a second-order tensor. They define stress equilibrium in a stress field. They have the form of differential equations but in practice they are used for the determination of changes in principal stresses by means of a process of integration. These equations are the foundation of all photoelastic analysis but they have found practically no place in classical stress analysis.

No principal-stress term appears independently in Equations (1) or (2). The principal stress differences, $(P - Q)$, $(Q - R)$, and $(R - P)$, however, do appear as independent quantities. It will be convenient in the discussion of stress-gradient problems to designate these independent gradient-determining terms by the new words *detend* for such terms as $(P - Q)$, and *double-detend* for such terms as $(P - Q)$, $(P - R)$.

Photoelasticians have introduced the word "isochromatic" to refer to *equidetend* lines in a stress field. This is a convenient and logical term for the person looking at a photoelastic color field, but it is neither logical nor convenient for any discussion of stress analysis. It would be very awkward to speak of the isochromatic of a point or of the isochromatic of P as being different from that for Q. As a matter of fact a photoelastic isochromatic does not exist in the stressed field or in the test member. It is created by the polarizing analizer for the recording instrument, the eye or camera. Let us therefore drop this artificial term which has little meaning in the stress field and accept a new term which is explicit in its meaning and application. Let us call these curves on which $P - Q$ has a constant value equidetend lines.

Lamé's rule, following his convention as to the sign of the radius of curvature, may now be stated more explicitly, especially with reference to the curvatures involved, as follows:

The gradient of a principal stress along its axis-line is equal to the sum of the ratios of its double detend to the respective radii of curvature of the projections of the other axis-lines on the mutual axis planes. A radius is positive for a gradient taken toward the center of curvature.

Imaginary Force-Component Equations of Stress Equilibrium.

Early researchers [2] developed the theory of elasticity as an analysis of the effects of combinations of imaginary force components.

The Cauchy expression for force equilibrium in a stressed body is:

$$\frac{\partial Xx}{\partial x} - \frac{\partial Xy}{\partial y} - \frac{\partial Zx}{\partial z} = 0,$$

$$\frac{\partial Xy}{\partial x} - \frac{\partial Yy}{\partial y} - \frac{\partial Yz}{\partial z} = 0, \tag{3}$$

$$\frac{\partial Zx}{\partial x} - \frac{\partial Yz}{\partial y} - \frac{\partial Zz}{\partial z} = 0.$$

Love, on page 126, names Xx, Yy, Zz, Xy, Yz, and Zx "components of stress," using the quotation marks to indicate that the definition is incomplete. On page 74 he defines the force acting across an area in a stressed body as "one aspect of a stress." On page 75 he defines "traction" as the "quantity of force per unit area." This definition will be used throughout this article. On page 81 he indicates a complete definition of stress by the following statement: "The stress at a point may be regarded as a single quantity related to direction: this quantity is not a vector, but has six components." These definitions are vague and unsatisfactory. More exact definitions can be obtained by a clear-cut differentiation between the real and the imaginary terms of Equations (1) and (3). This calls for an explicit differentiation between real and imaginary vector components.

Vector Forms.

Vectors are of two forms which may be conveniently identified as diffuse and ray. A diffuse vector, for example a field gradient, has a real value for any direction taken. The maximum value is merely a special case and is no more real than the others. A ray vector, for example a force or a velocity, has only one true value and direction. *Any components into which a ray vector may be analyzed are imaginary unless those components are given reality by facts.* The principal stresses P, Q, and R are equal to the forces they exert across unit normal area. They are similar to unit forces in all respects with this exception: A unit force is an independent physical entity while a principal stress is one member of a group of orthogonal principal stresses. The principal stresses should therefore be considered as ray components of the stress tensor. It should be considered that they have only one true value and only one true direction. Any component into which a principal stress may be divided should therefore be considered an imaginary component.

The terms of Equation (1) represent measurable physical entities. They should therefore be considered as real components. The terms of Equation (3), on the other hand, are not directly measurable; their value depends on the orientation of the coordinate system. They do not represent physical entities but rather the terms of a mathematical method of analysis. They also should therefore be considered as imaginary components.

The terms of Equation (1) are real and the terms of Equation (3) are imaginary. Either might be considered as either stress or force components. Figure 2 has been drawn to illustrate the difference in the steps followed in forming the terms of the two equations. In each case P is taken as plus 4 and Q as minus 3, and they are assumed to be at right angles to each other. For forming a term of Equation (1), $P - Q$ is equal to 7 and has the orientation of P. This is not vector addition and it should therefore be considered as a combination of stress components. The terms of Equation (3) are formed by the vector addition of imaginary components. If P and Q are combined by vector addition to form a term of Equation (3), $P - Q$ is equal to 5 and is diagonal to the right while $P + Q$ is still 5 but is diagonal to the left. $P + Q$, a sum of stress components, is equal to 1 and has the orientation of P. The rules for principal-stress additions or subtractions as used in Equation (1) apply only to companion principal stresses and are in no sense general. The rules for force-component addition or subtraction as used in Equation (3) are general and apply to all components. The terms of Equation (3), since they are formed according

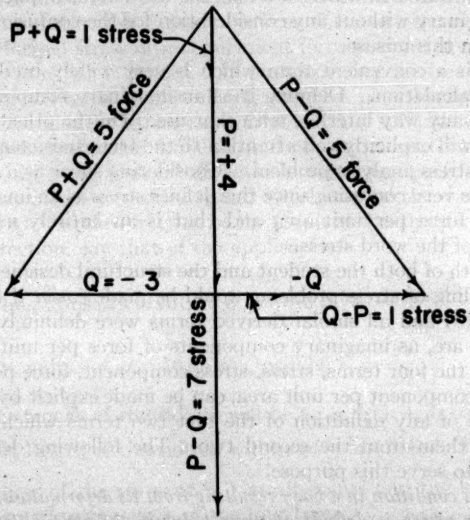

FIG. 2. Schematic diagram to illustrate the difference between a force vector addition or subtraction and combinations of normal principal stresses.

to a general rule applying to all force components, should be considered as imaginary force components, not as stress components.

There is confusion in comparing the works of different writers on stress problems, which is caused by differences in their definitions of the terms used. These differences can generally be explained by an explicit differentiation between real and imaginary terms. Many writers define stress as "force per unit area" and shear as "a parallel component of stress." These definitions have no meaning other than that of imaginary component of force per unit area. The word "stress" has been misused in order to avoid the distasteful implications of the word imaginary without any consideration for the confusion that might result from this misuse.

Shear is a convenient term which is very widely used in different forms of calculation. Defining it as an imaginary component of force will not in any way interfere with that use. On the other hand such a definition will explicitly call attention to the true character of the shear term in a stress analysis problem. Considering shear as a "component of stress" is very confusing since this defines stress as an imaginary component of force per unit area and that is an entirely unsatisfactory definition of the word stress.

The path of both the student and the structural designer to a better understanding of stress problems would be made easier if the terms of Equation (3) and all similar derived terms were definitely labeled for what they are, as imaginary components of force per unit area. The relation of the four terms, stress, stress component, force per unit area, and force component per unit area, can be made explicit by the general acceptance of any definition of the first two terms which clearly distinguishes them from the second two. The following definitions are suggested to serve this purpose.

Stress, a condition in a body resulting from its deformation, is a second-order tensor which is defined at every point in the body by its three components, the three orthogonal principal stresses with their orientations. A principal stress, P, Q, or R, is evaluated by the force it exerts across unit normal area.

If these or any other definitions which clearly distinguish between terms of stress and terms of force can receive general acceptance much confusion will be avoided.

PROOF OF SPECIAL STRESS-GRADIENT EQUATIONS.

The terms and data of photoelastic analysis have been defined. Equations (1) and (2) which are used by photoelasticians in stress analysis are special cases of the general principal-stress-gradient problem. They give gradients along stress-axis lines whereas a general solution will give the stress gradients in any direction. As a first step in developing general equations a simple proof of Equation (2) for plane stresses and of Equation (1) for three-dimensional stresses is given in the following paragraphs.

Proof of Equation (2) for Plane Stress.

Proof of Equation (2), which represents plane stress gradients taken in the direction of the principal stresses, has been discussed by several authors (3) and is the first step in the proof of the more complete Equation (1) for three-dimensional stress gradients. All analysis of stress effects must be made through the equivalent force components. Generally no distinction is made between these two terms. P may be either a principal stress or its equivalent force per unit area, depending on the way in which it is used. In the proof they are treated as forces.

Let *abcd* of Fig. 3a be a small segment of a plane-stress field of unit thickness bounded by stress-axis surfaces. Let *ab* and *cd* be parallel to the P principal stress while *bc* and *ad* are parallel to Q. Let the sides $ab = ds_1 = ad = ds_2 \doteq ds$ be very small compared to the radii of curvature of the two sides r_1 and r_2. Let α be the direction of the P axis and $\alpha - 90$ the direction of the Q axis at a with reference to some fiducial direction, say that of the applied load.

The radius of curvature of *ab* will be $r_1 = \dfrac{ds}{d(\alpha - 90)}$,

the radius of curvature of Q will be $r_2 = \dfrac{ds}{d\alpha}$,

the length of the side *bc* will be $bc = ds(1 + ds/r_2)$,

the length of the side *cd* will be $cd = ds(1 + ds/r_1)$.

The thickness of the segment is 1 at all points and the lengths of the edges are equal to the areas of the four faces.

Since the section *abcd* is in equilibrium the summation of all forces acting in any direction on the boundaries of the section must be zero. The forces across the sides *ab* and *ad* are respectively Qds and Pds. Considering now equilibrium of forces in the P orientation, the forces across *ad* and *bc* plus the component of Qds in the direction of P give the equation:

$$Pds + \frac{Qds^2}{r_2} - (P + dP)ds\left(1 + \frac{ds}{r_2}\right) = 0,$$

$$ds^2\frac{(Q - P)}{r_2} - dPds = 0.$$

Differentiation is away from the center and by definition r_2 is nega-

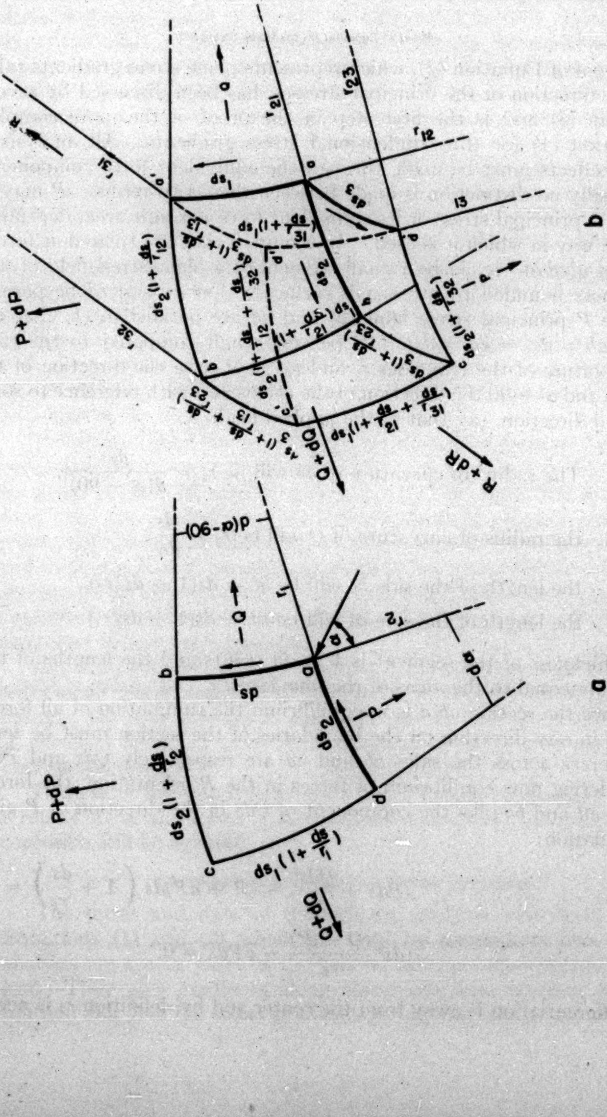
R
r_{31}
o'
ds_1
o
r_{21}
— Q
r_{23}
r_{12}
ε_{sp}
13
d
$ds_1(1+\frac{ds}{r_{31}})$
$\varepsilon_{sp}(1+\frac{ds}{r_{13}})$
d'
ds_2
b
— P
P + dP
r_{32}
$ds_2(1+\frac{ds}{r_{12}})$
$ds_2(1+\frac{ds}{r_{12}}+\frac{ds}{r_{32}})$
$(1+\frac{ds}{r_{21}})ds_1$
$\varepsilon_{sp}(1+\frac{ds}{r_{23}})$
b'
c
$ds_2(1+\frac{ds}{r_{32}})$
c'
$\varepsilon_{sp}(1+\frac{ds}{r_{13}}+\frac{ds}{r_{23}})$
Q + dQ
$ds_1(1+\frac{ds}{r_{21}}+\frac{ds}{r_{31}})$
R + dR
b
d(α-90)
— Q
r_1
r_2
α
b
a
ds_1
ds_2
— P
dα
P+dP
$ds_2(1+\frac{ds}{r_2})$
d
c
$ds_1(1+\frac{ds}{r_1})$
Q+dQ
a

FIG. 3. Schematic diagram to illustrate equilibrium of principal stresses.

tive. Therefore, we have the expression of Equation (2):

$$\frac{dP}{ds_1} = \frac{P - Q}{r_2}.$$

The same argument can be repeated for the gradient of Q.

Proof of Equation (1) for Three-Dimensional Stress.

Development of proof of the double-detend expression for three-dimensional stress gradients for Equation (1) is simply an extension of the argument used for the two-dimensional proof of Equation (2). Let the small segment of the three-dimensional stress field shown in Fig. 3b be bounded by stress-axis surfaces. Let

$$aa' = ds_1 = ab = ds_2 = ad = ds_3 = ds$$

be much smaller than the radii of curvature of the different edges.

Let r_{21} be the radius of curvature of the projection of the P stress axis upon the P, Q plane, and r_{31} the radius of curvature of its projection upon the P, R plane, with other radii having similar notation. Since the angles are very small, second-order effects can be neglected in computing the lengths of edges, areas of sides, and the force normal to the sides. The small angles through which an edge turns may be assumed to be equal to the length of the edge divided by the radius of curvature. The lengths of the different sides are shown in Fig. 3(b).

The summation of all forces acting on the body taken in any direction must be zero. Let us consider the forces acting in the P orientation. The areas of the normal faces are:

$$abcd = (1 + (1/2r_{32} + 1/2r_{23})ds)ds^2 = a_1,$$
$$a'b'c'd' = (1 + (1/r_{12} + 1/r_{13} + 1/2r_{23} + 1/2r_{32})ds)ds^2 = a_2.$$

The equation for the force acting in the P orientation is equal to the normal force across these two faces plus the component in the P direction of the two normal forces Qds^2 and Rds^2. This gives the equation:

$$Pa_1 - (P + dP)a_2 + \frac{Qds^3}{r_{12}} + \frac{Rds^3}{r_{13}} = 0,$$
$$ds^3(- P/r_{12} - P/r_{13} + Q/r_{12} + R/r_{13}) - dPds^2 = 0.$$

As with the plane stress, r is negative in the differentiation. This gives the expression in the form used in Equation (1)

$$\frac{dP}{dS} = \frac{P - Q}{r_{12}} + \frac{P - R}{r_{13}}.$$

Other parts of Equation (1) may be proven by a similar argument.

DEVELOPMENT OF THE MAXIMUM-STRESS-GRADIENT TRIANGLE AND OF THE GENERAL-PLANE-STRESS-GRADIENT EQUATIONS.

A general solution for stress gradients in a plane-stress field in static equilibrium is obtained by the following argument.

The Gradient and Intercepts Triangles.

The numbers which represent the magnitudes of the two principal stresses and their detend for any given plane-stress field are topographic quantities. That is, they can be represented by a continuous surface over the field. The slopes of these three surfaces would then represent the gradients of the three quantities P, Q, and $(P - Q)$. Each of the three gradients, taken in any direction from any given point, must be equal to some maximum gradient, indicated by Δ, times the cosine of the angle which the chosen direction makes with the direction of the maximum gradient, and the gradient of P must be equal to the gradient of $P - Q$ plus the gradient of Q for any chosen direction. It should be possible, then, to represent the maximum gradients for any point in a plane stress field by a triangle with the three sides proportional to ΔP, ΔQ, and $\Delta(P - Q)$, and having the same directions. This will be called the *gradient triangle*. The projection of the sides of the *gradient triangle* on any line through the point which determines it will give the gradients of P, Q, and $(P - Q)$ in the direction of the line at that point.

The equidetend, isoclinic, and stress-axis maps shown in Fig. 1 respectively by the dot dash, the dotted, and the orthogonal solid lines give the data required for a solution, by means of Equation (2), for the gradient triangle of any point in the field. One side of the triangle, $\Delta(P - Q)$, the detend gradient, is obtained directly from the equidetend map. It is normal to the equidetend lines and has a length given by the equation:

$$\Delta(P - Q) = \frac{d(P - Q)}{ds_4}, \tag{4}$$

where $d(P - Q)$ is the detend change between two near equidetends, and ds_4, Fig. 4, is the normal distance between them. Both quantities are positive.

Figures 4 and 5 illustrate the development of a graphic solution for the other two sides of the gradient triangle, ΔP and ΔQ, by means of the right-angled *intercepts triangle Oab*, whose legs ds_1 and ds_2 are short sections of P and Q axis lines taken from O in the direction of increasing P and Q values. The hypotenuse ab which forms the angle ω with the side ds_1 is a section of the isoclinic $\alpha + d\alpha$. The P axis line has the orientation α at the point O and $\alpha + d\alpha$ at a. Its radius of curvature r_1 is therefore numerically equal to $ds_1/d\alpha$. The Q axis line has the orientation $\alpha + 90$ degrees at O and $\alpha + d\alpha + 90$ degrees at b. Its radius of curvature r_2 is therefore numerically equal to $ds_2/d\alpha$.

The height of the *intercepts triangle*, ds_3, can always be measured much more accurately than the legs ds_1 or ds_2. Therefore generally these legs will be defined in terms of ds_3, which is taken with the same sign as $P - Q$. The P axis intercept ds_1 is numerically equal to $ds_3/\sin \omega$ and the Q intercept ds_2 is equal to $ds_3/\cos \omega$. The length r_3, which is defined as equal to $ds_3/|d\alpha|$, is not a radius of curvature but

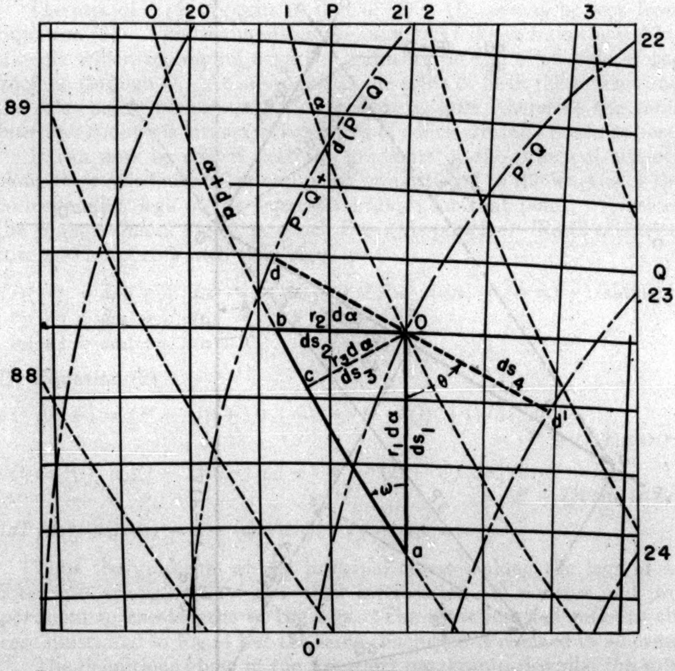

FIG. 4. A segment of a photoelastic map including stress axis, isoclinic, and equidetend lines, is drawn to illustrate the intercepts triangle which has sides proportional to the gradients of the principal stresses, P and Q_1 in their direction.

it will be found more convenient than r_1 and r_2 as they appear in Equation (2), in some stress gradient equations. Development of the *gradient triangle* involves a fourth length r_4 which is defined to be equal to $(P - Q)/\Delta(P - Q)$ or $(P - Q)ds_4/d(P - Q)$. It will also involve the angle θ between ds_1 and ds_4.

The sign of each angle and length described in the foregoing paragraphs must be determined before it can be used in stress-gradient equations. All angles are read clockwise; that is, a counterclockwise angle is considered negative. The angle α can be measured from any convenient base direction. It serves only to orient the P-stress-axis

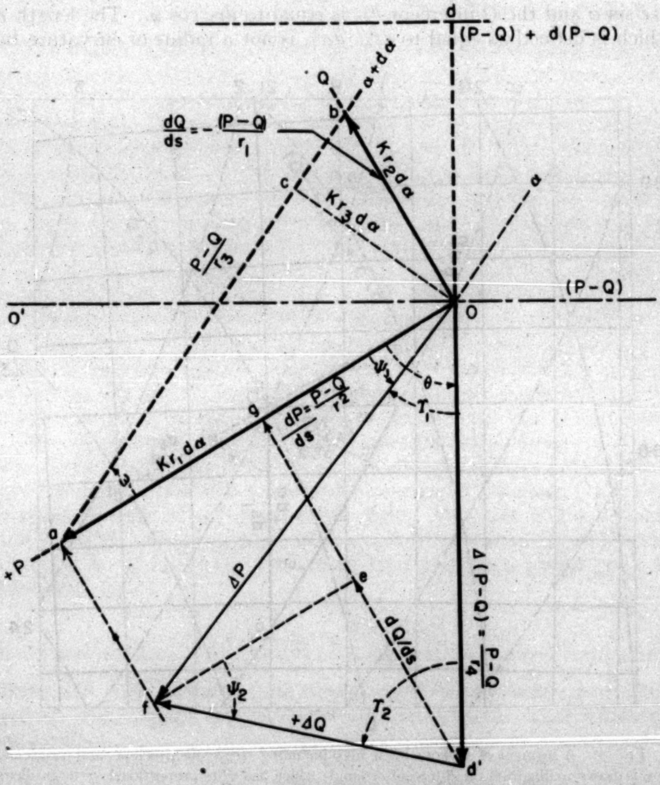
d
(P−Q) + d(P−Q)
α+dα
Q
b
dQ/ds = −(P−Q)/r₁
Kr₂dα
c
Kr₃dα
α
(P−Q)/r₃
O'
O
(P−Q)
ψ₁
θ
T₁
g
dP/ds = (P−Q)/r₂
Kr₁dα
ψ₃
a
+P
ΔP
e
dQ/ds
Δ(P−Q) = (P−Q)/r₄
ψ₂
f
+ΔQ
T₂
d'

FIG. 5. The intercepts triangle of Fig. 4 redrawn to the scale of the stress gradients dP/ds and $-dQ/ds$ for a graphic solution for the gradient triangle with the three maximum gradients ΔP, ΔQ and $\Delta(P - Q)$ for its sides.

line. All other angles are measured from the P-axis line and from the direction in which P increases. This is called the $+P$ direction.

The lengths ds_1 and ds_2 are considered negative or positive, depending on whether, from the point O, they are taken toward or away from the center of curvature of the perpendicular stress-axis line. An inspection of Fig. 4 and consideration of the relation of the curvatures to $d\alpha$ will show that as legs of an *intercepts triangle* one will always be positive while the other is negative, and that the associated r will always have the same sign as the ds used to determine it. From Equation (2) it follows that r_2 has the same sign as $P - Q$, while r_1 has the opposite sign. Thus by Equation (2) both gradients along the legs of an *intercepts triangle* have the same sign.

The sign of ds_1 is opposite to that of $(P - Q)$, as may be seen from Equation (2). This distinguishes the *intercepts triangle* from the similar triangle which represents negative gradients on the other side of the isoclinic through O. At any point O the signs of both the detend and the ds's change between the two stress axis lines; therefore the same *intercepts triangle* is formed independent of which stress is chosen as base.

It can now be shown that the gradients of the principal stresses along their axis lines are at any point proportional to the lengths of the corresponding legs of the *intercepts triangle* for that point. Consider the triangle Oab of Fig. 4, in which $P - Q$ is positive. Radii of curvature and intercepts with their signs are:

$$\begin{aligned} r_1(-) &= ds_1(-)/|d\alpha| = -\,ds_3(+)\sin|\omega|/|d\alpha| = -\,r_3(+)/\sin|\omega|, \\ r_2(+) &= ds_2(+)/|d\alpha| = r_3(+)/\cos\omega, \\ r_3(+) &= ds_3(+)/|d\alpha|. \end{aligned}$$

By Equation (2)

$$\begin{aligned} dP/ds(+) &= (P - Q)(+)/r_2(+) = (P - Q)(+)|d\alpha|/ds_2(+) \\ &= (P - Q)\cos\omega/r_3, \\ dQ/ds(+) &= (Q - P)(-)/r_1(-) = -\,(P - Q)|d\alpha|/ds_1(-) \\ &= (P - Q)\sin|\omega|/r_3, \\ (dP/ds)/(dQ/ds) &= [-\,(ds_1/ds_2)(+)] = \cot|\omega|. \end{aligned}$$

Thus the gradients of the principal stresses along the legs of an intercepts triangle away from their intersection are positive and proportional to the lengths of the legs. The signs inserted refer to the case illustrated in Fig. 4 but the same conclusion is reached in all cases.

The definitions given in the foregoing paragraphs describe the sides of the *intercepts triangle* in the terms of differential calculus. Integrations of principal stress change are made however, not by mathematical calculations for an infinitesimal triangle, but by measurements made on triangles of finite dimensions. In the following discussion the finite terms $\Delta\alpha$, DP, DQ, $D(P - Q)$, oa, ob, oc, and od will be used for actual numerical integrations or for any reference to finite terms.

Definitions of Terms.

The terms taken from photoelastic maps for use in the calculation of principal-stress gradients and illustrated in Figs. 4 and 5 are, for convenience, grouped as follows:

$P - Q$ is the detend of P and is equal to minus the detend of Q,
Δaob is the *intercepts triangle* in Fig. 4, or the similar *intercepts-gradient triangle* in Fig. 5,
$\Delta Ofd'$ is the *gradient triangle*,
P principal stress is chosen to be near the vertical in Fig. 4,
Q principal stress is chosen to be near the horizontal in Fig. 4,
$\Delta(P - Q)$ is the maximum gradient of $P - Q = \dfrac{d(P-Q)}{ds_4} = \dfrac{(P-Q)}{r_4}$,
ΔP is the maximum gradient of P,
ΔQ is the maximum gradient of Q.

The three maximum gradients ΔP, ΔQ, and $\Delta(P - Q)$ combine to form a *gradient triangle.*

$\dfrac{dP}{ds}$ is the positive gradient of P parallel to P,
$\dfrac{dQ}{ds}$ is the positive gradient of Q parallel to Q,
θ is the angle measured clockwise from the direction along a P-axis line in which P increases, designated $+P$, to the direction of $\Delta(P - Q)$. Variation is from 0 to plus or minus 180 degrees,
ω is the acute angle from $+P$ to an isoclinic line. In Fig. 5 ω is negative,
ψ_1 is the acute angle from $+P$ to the direction of ΔP,
ψ_2 is the angle from $+P$ to the direction of ΔQ. Variation is from 0 to plus or minus 180 degrees,
α is the direction angle of $+P$ measured clockwise from some fiducial direction. It characterizes an isoclinic,
ds_1, ds_2 and ds_3 are respectively the intercepts of P and Q stress-axis lines and the isoclinic normal between the two isoclinic lines for α and $\alpha + d\alpha$. ds_1 and ds_2 differ in sign; the one taken away from the centers of curvature of the perpendicular line is positive. Line ds_3 takes the sign of ds_2,
r_1 is the radius of curvature of the P stress-axis line and is equal to $ds_1/|d\alpha| = -r_3/\sin|\omega|$,
r_2 is the radius of curvature of the Q stress-axis line and is equal to $ds_2/|d\alpha| = r_3/\cos\omega$,
r_3 is equal to $ds_3/|d\alpha|$,
r_4 is equal to $(P - Q)/\Delta(P - Q)$ and takes the sign of $(P - Q)$.

Graphic Solution for the Gradient Triangle.

The *intercepts triangle Oab* of Fig. 4 is redrawn to the scale of the gradients dP/ds and dQ/ds, forming the *intercepts-gradient triangle Oab* in Fig. 5. The lengths of the two legs are found from Equation (2) as follows:

$$oa = \frac{dP}{ds} = \frac{(P-Q)}{r_2} = \frac{(P-Q)\cos\omega}{r_3} = \frac{(P-Q)|d\alpha|\cos\omega}{ds_3}, \qquad (5)$$

$$ob = \frac{+dQ}{ds} = -\frac{(P-Q)}{r_1} = \frac{(P-Q)\sin|\omega|}{r_3} = \frac{(P-Q)|d\alpha|\sin|\omega|}{ds_3}. \qquad (6)$$

The quantities $(P - Q)$, ω, $d\alpha$, and ds_3 are obtained by measurement. From these measurements a scale factor K, which gives the ratio between the legs of the *intercepts triangle* of Fig. 4 and gradients of the *intercepts gradient triangle* of Fig. 5, is obtained as follows:

$$Kds_1 = \frac{P-Q}{r_2},$$

$$K = -(P-Q)/r_1r_2d\alpha = (P-Q)d\alpha\sin 2|\omega|/2ds_3^2. \qquad (7)$$

The gradient triangle Ofd' of Fig. 5 is now constructed from the *intercepts gradient triangle Oab* in three steps as follows. The side Od' is drawn to represent $\Delta(P - Q)$, whose magnitude is given by Equation (4). Since dP/ds is equal to $\Delta P\cos\psi_1$ and dQ/ds is equal to $\Delta Q\sin|\psi_2|$ if $d'e$ is constructed parallel and equal to Ob then normals to Oa erected at a and to $d'e$ erected at e will intersect at f, the missing point of the *gradient triangle*, and Of will be proportional to ΔP and will lie in its direction, while $d'f$ will be proportional to ΔQ and will lie in its direction.

Two limiting conditions for values of ψ_1 and ψ_2 should be noted because they are necessary for a unique algebraic solution for these angles in terms of elements of the *intercepts triangle.* The angle ψ_1 is acute since it is one angle of a right triangle. The angle ψ_2 must always differ in sign from the angle ω. It is obvious from the construction of Fig. 5 that ΔQ and the isoclinic to which ω is measured lie on opposite sides of $+P$. It should be remembered that all of these developments are symmetrical with reference to the choice of P and Q; that is, it is immaterial which stress is chosen as the base for direction measurements or which detend is considered as positive.

Algebraic Development of the Stress-Gradient Triangle.

The graphic solution of the *gradient triangle* developed in Fig. 5 is simple, straightforward, and practically fool-proof. It is true, independent of the magnitudes of the vectors, and independent of the angles involved or the order in which they are taken. The same results can be obtained by an algebraic solution. In making this solution the signs of the different quantities involved may be very confusing because they are not obtained from a fixed coordinate system but rather from their use in an equation. The terms ΔP, ΔQ, $\Delta(P - Q)$, dP/ds, and dQ/ds as gradients are positive, independent of their absolute or relative direction. The terms $(P - Q)$, P, or Q may be either positive or negative and in any specific plane-stress problem, either of the two principal stresses may be taken as the control or P stress.

Using the notation of Fig. 5 the following equations are obtained.

$$\tan\psi_1 = \text{gradient of } P \text{ taken along } Q \text{ clockwise from } (dP/ds)/(dP/ds) = [\Delta(P - Q)\sin\theta \pm (dQ/ds)]/(dP/ds).$$

The sign before (dQ/ds) is to be taken positive or negative, depending on whether dQ/ds in the intercepts triangle is directed clockwise or counterclockwise by a negative or positive ω from (dP/ds); in either case $\pm(dQ/ds)$ is equal to $-(dP/ds)\tan\omega$. Hence

$$\tan\psi_1 = [(P - Q)\sin\theta/r_4 - (dP/ds)\tan\omega]/(dP/ds)$$

or

$$\tan\psi_1 = (r_3\sin\theta/r_4\cos\omega) - \tan\omega; \tag{8}$$

by Equation (2)

$$dP/ds = (P - Q)/r_2 = (P - Q)\cos\omega/r_3.$$

In a similar manner we may write:

$$\cot\psi_2 = \text{gradient of } Q \text{ taken along } P \text{ counterclockwise from } (dQ/ds)/(dQ/ds) = \pm[(dP/ds) - \Delta(P - Q)\cos\theta]/(dQ/ds).$$

The plus or minus sign ($\pm$) is to be taken opposite to the sign of ω; in any case $\pm(dP/ds)$ is equal to $-(dQ/ds)\cot\omega$. We then have:

$$\cot\psi_2 = -[(dQ/ds)\cot\omega \mp (P - Q)\cos\theta/r_4]/(dQ/ds).$$

Again the $\mp$ is taken opposite to the sign of ω and for both cases we have

$$\cot\psi_2 = (r_3\cos\theta)/(r_4\sin\omega) - \cot\omega; \tag{9}$$

by Equation (2)

$$dQ/ds = (P - Q)/r_1 = (P - Q)\sin|\omega|/r_3.$$

The lengths of the two sides of the gradient triangle are given by the expressions:

$$\Delta P = \frac{dP}{ds\cos\psi_1} = \frac{(P - Q)\cos\omega}{r_3\cos\psi_1}, \tag{10}$$

$$\Delta Q = \frac{+dQ}{ds\sin|\psi_2|} = \frac{(P - Q)\sin|\omega|}{r_3\sin|\psi_2|}. \tag{11}$$

In these expressions the ratios r_3/r_4 and $(P - Q)/r_3$ are always positive.

One side of the gradient triangle is given by Equation (4). The directions and lengths of the other two sides are given by Equations

(8), (9), (10), and (11). The five equations are based on five quantities, $(P - Q)$, θ, ω, r_3, and r_4, which can be independently measured for any point on an equidetend and isoclinic map. They do not involve the intercept gradients dP/ds and dQ/ds of Equation (2). Equation (4), together with any two of the other four equations, fully determines the gradient triangle. It is therefore doubly redundant and this double redundance can be used as a check on calculations or measurements made, or, two of the equations which cannot be solved accurately can be dropped and the solution can be based on the other equations which give more accurate results.

SOLUTIONS OF SPECIAL STRESS-GRADIENT PROBLEMS.

The general significance of the five Equations (4), (8), (9), (10), and (11) and their use in solving stress problems in statically stressed fields is illustrated by a number of special case solutions shown diagrammatically in Fig. 6. The notation is the same as for Figs. 4 and 5.

(a) When α, the direction angle of P or Q, reaches a maximum or minimum the P and Q stress-axis lines are straight, the gradient of α is zero, there is no direction for the isoclinic lines; therefore the angle ω and Equations (10) and (11) are indeterminate.

$$\begin{aligned} r_1,\ r_2 \text{ and } r_3 &= \infty \text{ by definition,} \\ \psi_1 &= 90 \text{ degrees by Equation (2),} \\ \psi_2 &= 0 \text{ degrees by Equation (2),} \end{aligned}$$

The gradient triangle is a right triangle with $\Delta(P - Q)$ as the hypotenuse, and

$$\begin{aligned} \Delta P &= \Delta(P - Q) \sin|\theta|, \\ \Delta Q &= \Delta(P - Q) \cos\theta. \end{aligned}$$

(b) When $(P - Q)$ is a maximum or minimum:

$$\begin{aligned} \Delta(P - Q) &= 0 \text{ by definition,} \\ r_4 &= \infty \text{ by definition,} \\ \psi_1 &= -\omega \text{ by Equation (8),} \\ \psi_2 &= -\omega \text{ by Equation (9),} \\ \Delta P = \Delta Q &= \frac{(P - Q)}{r_3} \text{ by Equations (10) and (11).} \end{aligned}$$

The gradient triangle is a straight line with ΔP equal to ΔQ and in the same direction.

(c) When P lies on an equidetend line:

$$\begin{aligned} \theta &= 90 \text{ degrees by definition,} \\ \psi_2 &= -\omega \text{ by Equation (9) and} \\ \Delta Q &= \frac{(P - Q)}{r_3} \text{ by Equation (11).} \end{aligned}$$

These relations complete the triangle.

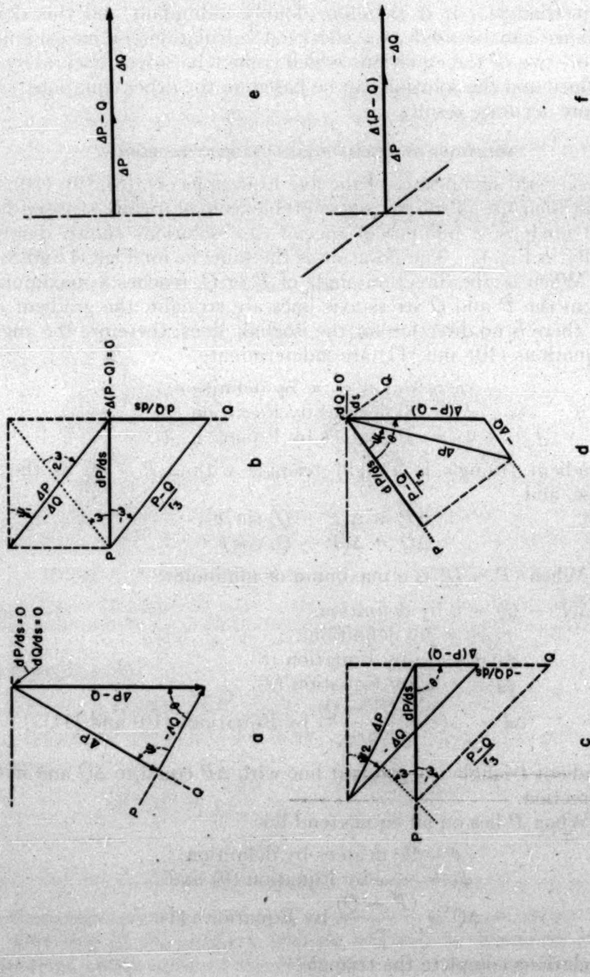
a
ΔP-Q
ΔP
-ΔQ
ψ
θ
P
Q
dP/ds=0
dQ/ds=0
b
Δ(P-Q)=0
dQ/ds
Q
ΔP
-ΔQ
ψ₁
2ω
dP/ds
ω
P
(P-Q)/r₃
c
Δ(P-Q)
-dQ/ds
Q
ΔP
dP/ds
-ΔQ
θ
ψ/2
ω
P
(P-Q)/r₃
d
dQ/ds=0
Q
Δ(P-Q)
-ΔQ
ΔP
θ
ψ
dP/ds
(P-Q)/r₁
P
e
ΔP
ΔP-Q
-ΔQ
f
ΔP
Δ(P-Q)
-ΔQ

FIG. 6. Special case and special field solutions of the stress gradient triangle problem.

(d) When P lies on an isoclinic line:

$$\left.\begin{aligned} r_1 &= \infty \\ \omega &= 0 \text{ degrees} \\ r_2 &= r_3 \end{aligned}\right| \text{ by definition,}$$

$\psi_2 = 0$ or 180 degrees, since, by Equation (2), $dQ/ds = 0$,

$$\frac{dP}{ds} = \frac{(P - Q)}{r_3} \text{ by Equation (2).}$$

STRESS GRADIENTS IN SPECIAL STRESS FIELDS.

An Infinite Plane Stressed by Radial Pressure on the Wall of a Circular Opening.

When an infinite sheet is stressed by normal force on the edge of a circular opening the stress field is symmetrical. P is assumed to be radial and Q tangential, r_1 is infinite, r_2 is equal to r_3 as in Case (d), and both are numerically equal to r, the distance from the center of the opening. ΔP, ΔQ and the isoclinic lines are radial, while the equidetends are tangential. The *gradient triangle is a radial straight line.* θ, ω, ψ_1, and ψ_2 are all 0 degrees.

By Equation (10), ΔP is equal to $\dfrac{P - Q}{r}$. Equation (11) is indeterminate but the value of ΔQ can be determined as equal to $-\Delta P$ by the following argument.

Since by symmetry the only action across any circular boundary centered on the center of the opening will be a normal pressure, a circular ring of any desired outer radius can be cut from the field and its action on the outer part of the field replaced by a normal pressure which will maintain the original stress pattern. By the rules of similitude the ratio Q/P at the edge of a circular opening in an infinite plane stressed normal by pressure on the edge of the opening must be independent of the radius of the opening. Therefore over the entire field the ratio Q/P must be constant. Then $aP = Q$; $P - Q = P(1 - a)$ and $P + Q = P(1 + a)$.

By Equation (2) for integration away from the center of the opening we have $dP/dr = -(P - Q)/r = -P(1 - a)/r$; $dP/P = -(1 - a) \times dr/r$, then $\log P = -(1 - a) \log r + C$.

$$P = Kr^{a-1} \qquad (12)$$

The discussion up to this point has been based on properties of equidetend or principal-stress-difference maps. The detend has directional properties which are a part of most stress-gradient problems. The sums of the principal stresses are nondirectional since at any point in a plane-stress field, the sum of two perpendicular force components is a constant independent of the direction in which the components are taken. This non-directional nature of the stress-sum map of a stress field simplifies the properties of that field. Any mathematical analysis applying to a field which can be fully described by a number for any point in the field will apply to a stress-sum map. In particular the second derivative, Laplace equation, $\nabla^2 = 0$ must be satisfied by any plane stress-sum map. Frocht (4) gave proof of this necessity. Coaker and Filon [3] on page 164 give the form of the equation as applied to a stress-sum map expressed in terms of polar coordinates where $P + Q$ is equal to V.

$$\nabla^2 = \partial^2 V/\partial r^2 + 1/r\, \partial V/\partial r + \partial^2 V/\partial \theta = 0.$$

The field being considered is symmetrical with respect to θ and the last term drops out. Substituting $P(1 + a)$ for V we have

$$\nabla^2 = (1 + a)(\partial^2 P/\partial r^2 + \partial P/r\partial r) = 0.$$

This equation is satisfied only when $a = -1$.

If P_0 is assumed to be the hydrostatic pressure on an opening of unit radius we then have from (12):

$$P = P_0 r^{-2} = -Q \text{ or } P + Q = 0 \tag{13}$$

and the gradients are given by the equation

$$\Delta P = -\Delta Q = \frac{P - Q}{r} = 2\frac{P_0}{r^3}. \tag{14}$$

The gradient triangle for this case is shown in *e* of Fig. 6.

Tangential Stresses in a Tube with Internal or External Pressure.

The radial and tangential stresses in a tube with either thick or thin walls can be determined by a simple extension of Equation (13) plus the assumption that a stress pattern is not changed by hydrostatic pressure or tension. Physically it is not possible to apply a hydrostatic tension. That, however, does not concern the mathematician.

It is desired to find the tangential stresses in a tube with inner radius r_1, and outer radius r_2 stressed by internal hydrostatic pressure P_x.

Consider a ring section of an infinite sheet stressed by a pressure P_0 on the walls of the unit circle. At the inner boundary of the ring, by Equation [13]

$$P_1 = -Q_1 = \frac{P_0}{r_1^2};$$

at the outer boundary of the ring the stresses are

$$P_2 = -Q_2 = \frac{P_0}{r_2^2}.$$

If we are to duplicate the stress pattern in the tube walls resulting from internal pressure P_x and no external pressure then

$$P_1 - P_2 = P_x = \frac{P_0}{r_1^2} - \frac{P_0}{r_2^2}\text{; or } P_0 = \frac{P_x r_1^2 r_2^2}{r_2^2 - r_1^2}.$$

From this it follows that

$$P_1 = \frac{P_x r_2^2}{r_2^2 - r_1^2} = -Q_1 = P_2 - P_x \tag{15}$$

and

$$P_2 = \frac{P_x r_1^2}{r_2^2 - r_1^2} = -Q_2. \tag{16}$$

The external radial pressure on the tube may be reduced to zero by applying a hydrostatic tension equal to P_2 to all elements. This reduces the inner radial pressure to P_x, the initial condition of the problem.

The external, Qt_2, and internal, Qt_1, tangential tension for a tube of inner and outer radii respectively r_1 and r_2, stressed by internal hydrostatic pressure P_x, are then given by the following equations:

$$Qt_2 = -2P_x \frac{r_1^2}{r_2^2 - r_1^2}, \tag{17}$$

$$Qt_1 = Qt_2 - P_x. \tag{18}$$

If a hydrostatic tension equal to P_1 is applied to all elements of the ring the internal radial pressure will be zero while there will be an external tension of $-P_x$. The tangential tensions in the ring will then be equal to the tangential compressions in a tube under external compression.

The internal, QC_1, and external, QC_2, tangential compressions in a ring of inner and outer radii respectively r_1 and r_2, stressed by an external hydrostatic compression P_x, are given by the following equations:

$$QC_1 = 2P_x \frac{r_2^2}{r_2^2 - r_1^2}, \tag{19}$$

$$QC_2 = QC_1 - P_x. \tag{20}$$

It should be remembered that P and Q are of opposite sign for Equation (18) and add while they are the same sign for (20) and subtract.

An Infinite Solid Stressed by Hydrostatic Pressure on the Wall of a Spherical Inclusion.

Consider the case of a spherical cavity in an infinite solid. Let P be radial and Q and R be the tangential principal stresses; then r_1 is infinite; r_2 and r_3 are equal to r, the distance from the center of the opening. By symmetry Q is equal to R and ΔP, ΔQ, and ΔR are all radial. As for plane stresses

$$\theta = \omega = \psi_1 = \psi_2 = \psi_3 = 0.$$

By Equation (1)

$$\Delta P = -2\frac{(P - Q)}{r}.$$

The argument for symmetrical relations between r, P, Q, and R is the same as for plane stresses and we have

$$P = \frac{Q}{a} = \frac{R}{a}$$

and

$$P - Q = P - R = P(1 - a) = \text{and } P + Q = P + R = P(1 + a).$$

Three-dimensional gradient equilibrium by Equation (1) is

$$\frac{dP}{ds} = -2P\frac{(1 - a)}{r} \text{ or } \frac{dP}{P} = -2(1 - a)\frac{dr}{r}.$$

$$\operatorname{Log} P = -2(1 - a) \quad \operatorname{Log} r + c \text{ or } P = Kr^{2a-2} = \frac{Q}{a}. \tag{21}$$

Three-dimensional-stress sums, like the two-dimensional sums, are constant and must satisfy Laplace's equation for any plane field as is shown by Love [2] on page 135. Since Q is equal to R, $P + Q + R$ can be written $P(1 + 2a)$ and the Laplace equation for a plane formed by P and Q axis lines becomes:

$$\nabla^2 = (1 - 2a)\left(\frac{\partial^2 P}{\partial r^2} + \frac{\partial P}{r \partial r}\right) = 0.$$

This equation is satisfied when $a = -0.5$. Again if P_0 is assumed to be the hydrostatic pressure on an opening of unit radius which is required to produce a given stress pattern we have from (21):

$$P = -2Q = -2R = \frac{P_0}{r^3}, \tag{22}$$

P being radial and Q and R tangential.

Gradients of P, Q and R are given by the equation,

$$\Delta P = -2\Delta Q = -2\Delta R = 2\frac{(P - Q)}{r} = 3\frac{P}{r} = 3\frac{P_0}{r^4}. \tag{23}$$

The gradient triangle for the P, Q plane is shown in Fig. 6f.

Tangential Stresses in a Spherical Shell Stressed by Internal or External Hydrostatic Pressure.

The tangential stress components for a spherical shell of inner and outer radii r_1 and r_2 stressed by internal pressure P_x can be calculated from Equation (22) as follows:

Assume an infinite solid stressed by internal pressure P_0 on the walls of a spherical opening of unit radius until the difference between radial pressures at r_1 and r_2 is equal to P_x.

The stress at the inner shell surface will be:

$$P_1 = \frac{P_0}{r_1^3} = -2Q_1.$$

The stress at the outer surface will be

$$P_2 = \frac{P_0}{r_2^3} = -2Q_2;$$

then

$$\frac{P_0}{r_1^3} - \frac{P_0}{r_2^3} = P_x$$

or

$$P_0 = P_x \frac{r_1^3 r_2^3}{r_2^3 - r_1^3},$$

$$P_1 = P_x \frac{r_2^3}{r_2^3 - r_1^3} = P_2 - P_x = -2Q_1,$$

$$P_2 = P_x \frac{r_1^3}{r_2^3 - r_1^3} = P_1 - P_x = -2Q_2.$$

The segment of the infinite solid between r_1 and r_2 will have the stress components of the spherical shell stressed by internal pressure P_x if the infinite solid is subjected to a hydrostatic tension equal to $-P_2$. The internal and external radial stresses will be reduced to P_x and 0 respectively, the initial condition of the problem, and the tangential stresses will be given as follows.

Letting Qt_1 and Qt_2 respectively represent the internal and external tangential stresses of a spherical shell of internal and external radii r_1 and r_2 stressed by internal pressure P_x, we have

$$Qt_2 = -\tfrac{3}{2} P_x \frac{r_1^3}{r_2^3 - r_1^3} \tag{24}$$

and

$$Qt_1 = Qt - \frac{P_x}{2}. \tag{25}$$

If the infinite solid is subjected to a hydrostatic tension equal to $-P_1$ the external and internal radial stresses will be $-P_x$ and 0 respectively. The tangential tensions will be equal to the compressions in a spherical shell subjected to external compression and may be stated as follows.

Letting QC_1 and QC_2 respectively represent the internal and external tangential compressions of a spherical shell of internal and external radii r_1 and r_2 stressed by external hydrostatic pressure P_x, then

$$QC_1 = \tfrac{3}{2}P_x r_2^3/(r_2^3 - r_1^3) \tag{26}$$

and

$$QC_2 = QC_1 - \frac{P_x}{2}. \tag{27}$$

The equations for tangential stresses in cylindrical and spherical shells stressed by internal or external pressure are given by Timoshenko [5] in his Articles 22 and 102 as special cases of the general solution:

$$\phi = A \log r + Br^2 \log r + Cr^2 + D. \tag{28}$$

The physical significance of the terms of this solution are not evident. A solution where the meaning and relation of the different terms are known, as they are for Equations (4) and (8) to (11), may have some advantage. This would be especially true for cases where a complete solution was impossible and an approximate solution had to be accepted. These illustrations and developments are of special interest because of the many cases where a designer must extrapolate calculations into regions of high stress concentration where measurements or exact calculations cannot be made. Very often the stress pattern in the neighborhood of such a region will closely approximate the conditions of one of these special cases of stress gradients.

INTEGRATION OF CHANGES IN P AND Q BY MEANS OF THE STRESS GRADIENT TRIANGLE.

The projections of the sides of a *gradient triangle* on a line drawn in any direction through its source give the gradients of P, $-Q$, and $P - Q$ in the direction of the line. Integrations of these changes can be made by the following rule:

The integrated changes of the principal stresses P and $-Q$ and the detend $(P - Q)$ along any line in a stress field are proportional to the areas swept out along the line by the normal projections of the sides of the *gradient triangle* rotated through 90 degrees.

Integration by this rule can be made graphically by plotting the line of integration straight independently of its curvature on the field. This is illustrated in Figs. 7 and 8 where integrations for the lines 0–10.75 and 5.5–14.7 of Fig. 1 are plotted along straight lines.

Integrations can be made in any direction, but the plotting is simpler along or normal to the equidetend lines because the direction of $\Delta(P - Q)$ is fixed. Generally integrations parallel to $\Delta(P - Q)$ offer the easiest approach to values of the principal stresses at the points of greatest stress concentration. These values are usually the goal of the stress analysis.

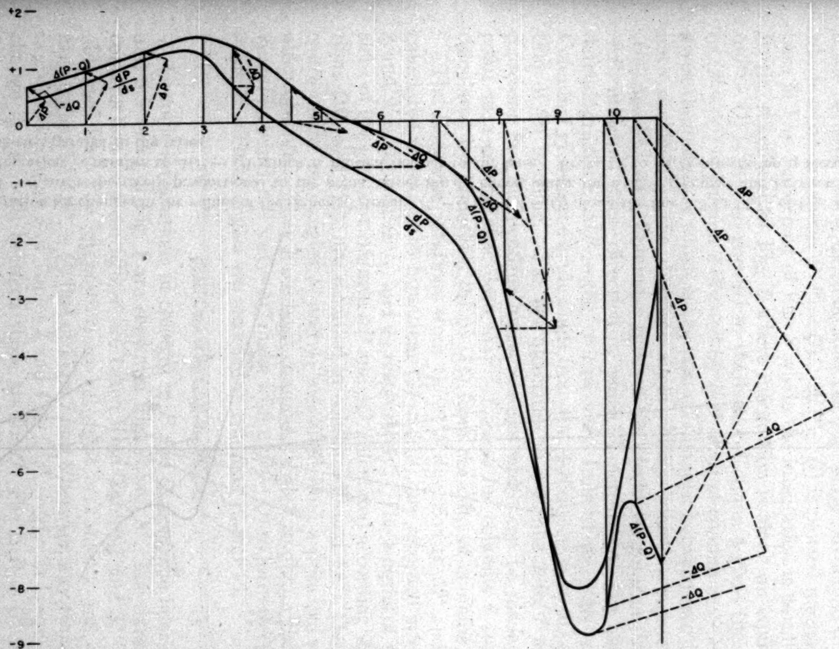

FIG. 7. Integration for changes in the value of the principal stresses P, $-Q$ and $(P - Q)$ along the line O to 10.75 of Fig. 1. The change in P is proportional to the area under the dP curve. The change in $(P - Q)$ is proportional to the area under the $\Delta(P - Q)$ curve. The change of $-Q$ between any two stations is of course proportional to the area between the two curves over the base between the two stations. The integration is parallel to $\Delta(P - Q)$ which is plotted normal to the base throughout.

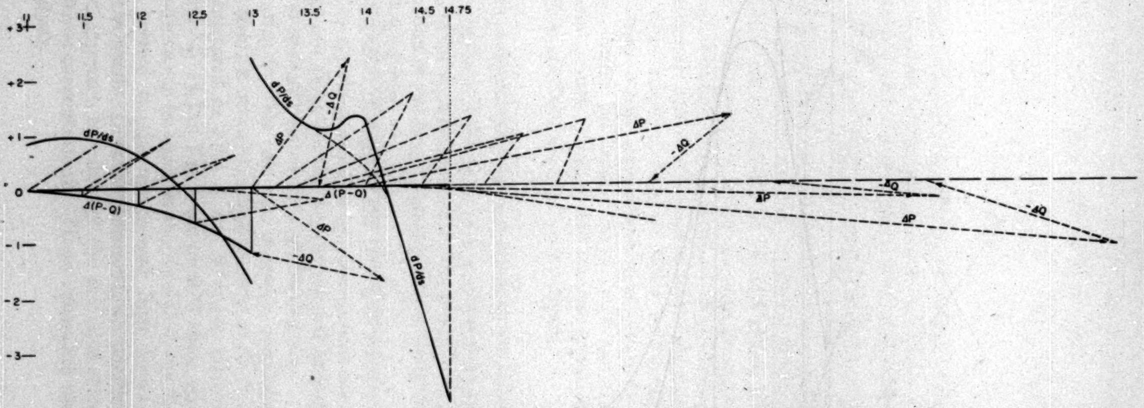
11
11.5
12
12.5
13
13.5
14
14.5
14.75
+3
+2
+1
0
-1
-2
-3
dP/ds
dP/ds
dP/ds
Δ(P-Q)
Δ(P-Q)
dP
dP
-ΔQ
-ΔQ
-ΔQ
-ΔQ
ΔP
ΔP
ΔP
ΔQ

FIG. 8. Integration for changes in the values of the principal stresses P, $-Q$ and $(P-Q)$ along the line 5.5 to 14.75 of Fig. 1. The changes in P, $(P-Q)$ and $-Q$ are respectively proportional to the areas, under the dP curve, under the $\Delta(P-Q)$ curve and between the two curves. From 5.5 to 13 integration is parallel to $\Delta(P-Q)$ which is plotted normal to the base. From 13 to 14.75 integration is along an equidetend and $\Delta(P-Q)$ is plotted parallel to the base.

Graphic Integration Data.

Data from Fig. 1 for the graphic integrations of Figs. 7 and 8 are recorded in Tables 1 and 3 where column 1 gives the stations and column 2 gives the values of the detends $(P - Q)$. Column 3 gives the values of the detend gradient $\Delta(P - Q)$ obtained by dividing the normal distance between two equidetend lines *Od*, Fig. 4, by the detend difference which separates them, $D(P - Q)$. $\Delta(P - Q)$ is plotted normal to the base in Fig. 7 and part of Fig. 8 where the integration is parallel to it. In the remainder of Fig. 8 where integration is along an equidetend, $\Delta(P - Q)$ is plotted parallel to the base. Column 4 gives the ratio r_4 or $(P - Q)$ of column 2 divided by $\Delta(P - Q)$ of column 3. Column 5 gives the radius r_3 which is the normal distance between two isoclinic lines divided by the direction-angle change which separates them $(OC/\Delta\alpha$, Fig. 4). Column 6 gives θ the clockwise rotation from the $+P$, principal stress direction in which P increases, to the maximum detend gradient, $\Delta(P - Q)$. θ can vary from 0 to -180 degrees. Column 7 gives ω the acute angle from the $+P$ direction to the orientation of an isoclinic. The six quantities of columns 2 to 7 inclusive represent five independent measurements which can be made at any point in a photoelastic data map. Figures for columns 8 to 13 are calculated by Equations (8) to (11) from the data of columns 2 to 7.

The clockwise rotations from the $+P$ direction to ΔP and $-\Delta Q$, calculated according to Equations (8) and (9), are recorded in columns 8 and 9 as the angles ψ_1, and ψ_2 respectively. The lengths of the two sides of the gradient triangle ΔP and ΔQ, calculated according to Equations (10) and (11), are recorded in columns 10 and 11. The significance of the two angles ψ_1 and ψ_2 measured from a principal stress is often obscure. They have been combined with θ to obtain the angles τ_1 and τ_2, which ΔP and $-\Delta Q$ make with $\Delta(P - Q)$ to form the *gradient triangle*. These angles are recorded in columns 12 and 13.

Construction of the Integration Figures.

Graphic integrations of changes in P and Q along the lines 0–10.75 and 5.5–14.75 of Fig. 1 are shown in Figs. 7 and 8. The data for the construction of these figures are taken from Tables 1 and 3. Results obtained from the integration are recorded in Tables 2 and 4. The area under the $\Delta(P - Q)$ line of Figs. 7 and 8 between any two stations is given in column 3 of Tables 2 and 4 as $D(P - Q)$. It is equal to the change in $(P - Q)$ between those stations. The $\Delta(P - Q)$ lines are drawn from values given in columns 3 of Tables 1 and 3. Values of the three quantities $(P - Q)$, $D(P - Q)$, and $\Delta(P - Q)$ along a line of integration can be determined much more accurately by an adjustment of all three values to fit than they can be measured separately. This adjustment added one significant figure to each value.

Tables 1 and 2 and Fig. 7 show an integration parallel to $\Delta(P - Q)$ along line 0–10.75 of Fig. 1. Values of $\Delta(P - Q)$ from column 3 of Table 1 were plotted normal to the base in the light solid lines to form the first side of the *gradient triangle.* The two broken lines forming the other sides were plotted with lengths taken from columns 10 and 11, or directions taken from columns 12 and 13. Four quantities evaluated by Equations (8) to (11) are given for each triangle, while only two are needed for plotting. Preference in each case was given to the two quantities, either length or angle, which showed the least variation with variation of the measured quantities by which they were determined. For example at Stations 3.5 and 4.5 errors in the measurement of θ which is near 90 degrees leave the cosine near zero and Equation (9) is more accurate than Equation (8). The value ψ_2 equal to $-\omega$ was accepted for these stations. This relation of the two angles made Equation (11) less sensitive to errors in measurement than Equation (10) and for these stations values of ΔQ were accepted to complete the gradient triangle. Gradient triangles plotted from these data are shown

TABLE 1.

Stress Data Measured from Fig. 1 and Calculated by Equations (8) to (11) and Used in the Graphical Integration of the Principal Stresses along the Line 0 to 10.75, Shown in Fig. 7.
The results are tabulated in Table 2.

Station	$P-Q$	$\Delta(P-Q)$	r_t	r_s	θ	ω	ψ_1	ψ_2	ΔP	$-\Delta Q$	τ_1	τ_2
	lbs./in.²		in.		degrees				lbs./in.²		degrees	
0	17.8	.62	28.6		−126	~	−90	0	.50	.36	36	126
1	18.7	.90	20.8		−118	~	−90	0	.79	.42	28	118
2	19.6	1.25	15.7	100	−108	~	−90	0	.95	.31	18	108
3	21.0	1.5	14.0	43	− 90	−72						
3.5	21.5	1.3	16.5	28	− 90	−61	− 59	61	.72	.77	31	151
4	22.3	.95	23.8	22	− 90	−50		50				
4.5	22.6	.60	37.7	18.8	− 90	−40	11	40	.93	1.20	101	130
5	22.9	.27	85	16.8	~	−30						
5.5	23	* .00	∞	15.2	~	−29	32	32	1.52	1.52	~	~
6	22.9	− .22	104	14	~	−29						
6.5	22.7	− .40	57	13	83	−28						
7	22.5	− .70	32	12	81	−28	43	30	2.3	1.77	−38	− 51
7.5	22.1	−1.11	20	11	79	−27						
7.8	21.6	−1.81	12	11	81	−26						
8.1	20.9	−2.93	7.1	14.2	83	−25	69	33	26.2	1.14	−14	− 50
8.5	19.5	−4.5	4.3	30	85	−24						
8.8	17.8	−7.0	2.5	100	90	−12	+90	12	7.0	0	~	~
9.1	15.4	−8.3	1.82	∞	− 82	12						
9.5	12.0	−9.0	1.34	100	− 72	25	−89	−179		2.8	−17	−106
9.8	9.3	−8.5	1.10	12	− 65	29	−85	−172.6	6.7	2.87	−20	−107.6
10.1	7.0	−6.7	1.01	4.2	− 57	33						
10.3	5.7	−6.7	.85	2.3	− 40	38	−72	−156	6.3	3.7	−32	−116
10.5	4.3	−7.3	.59	1.8	− 30	44						
10.7	2.8	−7.7	.36	1.0	− 10	50	−62	−159.5	3.90	6.18	−52	−149.5
10.75	2.4											

Stress data for integration parallel to $\Delta(P - Q)$.

* Beyond 5.5 integration is opposite to the gradient $\Delta(P - Q)$.

in the figure. The $\Delta(P - Q)$ and the $\frac{dP}{ds}$ lines outline areas swept out by the normal projections of the $\Delta(P - Q)$ and the ΔP sides of the gradient triangle as it is moved along the line of integration. The area swept out by ΔQ is of course equal to the area between these two lines. The area under the line marked $\frac{dP}{ds}$ between any two stations is proportional to the change in P between those stations and is recorded as

TABLE 2.

Integrated Values of P and −Q Principal Stresses along the Line 0 to 10.75 of Fig. 1.
The values are taken from the areas under the curves of Fig. 7.

Station	$P-Q$	$D(P-Q)$	DP	$-DQ$	P	$-Q$
0	17.86	.74	.52	.22	18.86	−1.0
1	18.60	1.06	.90	.16	19.38	− .78
2	19.66	1.39	1.19	.20	20.28	− .62
3	21.05	1.27	.65	.62	21.47	− .42
4	22.32	.62	− .17	.79	22.12	.20
5	22.94	.06	− .33	.39	21.95	.99
5.5	23.00	− .06	− .47	.41	21.62	1.38
6	22.94	− .43	−1.41	.98	21.15	1.79
7	22.51	− .44	−1.06	.62	19.74	2.77
7.5	22.07	− .44	− .83	.39	18.68	3.39
7.8	21.63	− .71	−1.06	.35	17.85	3.78
8.1	20.92	−1.48	−1.92	.44	16.79	4.13
8.5	19.44	−1.68	−1.90	.22	14.87	4.57
8.8	17.76	−2.33	−2.28	−.05	12.97	4.79
9.1	15.43	−3.50	−3.26	−.24	10.69	4.74
9.5	11.93	−2.63	−2.38	−.25	7.43	4.50
9.8	9.30	−2.28	−2.07	−.21	5.05	4.25
10.1	7.02	−1.40	−1.11	−.23	2.98	4.04
10.3	5.62	−1.34	− .89	−.51	1.87	3.81
10.5	4.28	1.50	− .66	−.84	.98	3.30
10.7	2.78	38	− .13	−.25	.32	2.46
10.75	2.40				.19	2.21
1	2	3	4	5	6	7

Integrated stress values parallel to $\Delta(P - Q)$.

DP values in column 4 of Tables 2 and 4. They are summed for P values in column 6. The area between the lines $\frac{dP}{ds}$ and $\Delta(P - Q)$ between any two stations is proportional to the change in $-Q$ between those stations and is tabulated in column 5 as $-DQ$. They are summed for $-Q$ values in column 7.

The summations of columns 6 and 7 start in the first line at Station 0 with the assumption that Q at this point was a tension of 1. Station 10.75, the last line of the table, is on the edge of the opening. P at this point is normal to the edge and is therefore zero. The value 0.19 shown in the table represents an error well within the limit of accuracy, about ±1, for the work.

TABLE 3.

Stress Data for Integration Normal to $P - Q$.

Stress data measured from Fig. 1 and calculated by Equations (8) to (11) and used in the graphical integration of the principal stresses along the Line 5.5 to 14.75 as shown in Fig. 8. The results are tabulated in Table 4.

Station	$P-Q$	$\Delta(P-Q)$	r_4	r_3	θ	ω	ψ_1	ψ_2	ΔP	$-\Delta Q$	τ_1	τ_2
	lbs./in.^{3}		in.		degrees				lbs./in.^{3}		degrees	
5.5	23.0	0		15.2	155	−29	~	~	1.52	1.52	~	~
11.5	23	.1	230	14.2	144	−22	24	18	1.75	1.87	−120	−126
12	22.9	.3	72	12.8	130	−15	26	12	1.83	2.08	−104	−118
12.5	22.7	.65	35	10.6	107	− 8	24	6	2.30	2.34	− 83	−101
13	22.2	1.2	18.5	8.4	79	− 2.5	26	~	2.93	2.40	− 53	~
13.2	22.2	1.6	14.9	8.0	57	1	23	~	3.04	1.90	− 34	~
13.4	22.2	2.2	10.7	7.2	43	4	21	− 4	3.34	3.1	− 22	− 47
13.6	22.2	2.9	7.7	6.2	32	6	18	−24	3.74	.84	− 14	− 56
13.8	22.2	3.8	5.9	4.8	20	11	5	−40	4.5	1.5	− 15	− 60
14	22.2	5.0	4.4	3.3	9	8	− 2	−29	6.6	2.1	− 11	− 38
14.2	22.2	6.7	3.3	2.3	− 2	6	4	− 3	9.7	18.2	− 6	− 1
14.4	22.2	9.5	2.4	1.7	−11	− 1.5	− 6	5	13.2	38.8	5	16
14.6	22.2	14.3	1.6	1.2	−20	− 4	−11	13	18.4	5.4	10	37
14.75	22.2											
1	2	3	4	5	6	7	8	9	10	11	12	13

Discussions of Integration Parallel to $\Delta(P - Q)$.

The test member from which the maps of Fig. 1 were drawn was a pyralin sheet 12 inches wide by 22 inches long with a circular hole 4 inches in diameter in the center. The load was applied as a uniform extension of the ends. The direction of the principal stress, P, near the edge of the member must be parallel to the edge, or 0 degrees. It is also 0 degrees at the midsection. The slope at Station 0 is 86 degrees. For integration along a Q axis line from the edge to Station 0, ds_2 will be negative and since $Q - P$ is also negative, dQ by Equation (2) will be positive. It was estimated that the increase in Q from the edge where it was zero to Station 0 was equal to 1.0 or a little more than 5 per cent. of the P stress which was placed as 18.86 by this estimate.

This estimation of the minor principal stress at a non-critical point is included as a part of the integration because very often in a photo-elastic stress investigation it is possible to estimate the value of the minor stress at some convenient point with all of the accuracy required for the total study. Integration from that point into more critical regions has then all of the accuracy required of the problem. In this test member P is the dominant stress over most of the area and since all stress-gradient equations include the detend $P - Q$, values of $-Q$, $-\Delta Q$, and $-DQ$ are recorded in the tables. The estimated Q tension is therefore recorded in the first line of column 7, Table 2 as -1 in the column headed $-Q$.

The performance of a stress integration along any line, like most operations in calculus, will call for special treatment to meet the pe-

TABLE 4.

Integrated Stress Values Normal to (P − Q).

Integrated values of P and $-Q$ principal stresses along the Line 5.5 to 14.75 of Fig. 1. The values are taken from the areas under the curves of Fig. 8.

Station	$P-Q$	$D(P-Q)$	DP	$-DQ$	P	$-Q$
5.5	23	−.02	.44	−.46	21.62	1.38
11.5	22.98	−.10	.38	−.48	22.06	.92
12	22.88	−.22	.10	−.32	22.44	.44
12.5	22.66	−.46	−.45	−.01	22.54	.12
13	22.2	−.00	.39	−.39	22.09	.11
13.2	22.2	−.00	.28	−.28	22.48	− .28
13.4	22.2	−.00	.22	−.22	22.76	− .56
13.6	22.2	−.00	.21	−.21	22.98	− .78
13.8	22.2	−.00	.25	−.25	23.19	− .99
14	22.2	−.00	.11	−.11	23.44	−1.24
14.2	22.2	−.00	−.24	.24	23.55	−1.35
14.4	22.2	−.00	−.44	.44	23.31	−1.11
14.6	22.2	−.00	−.41	.41	22.87	− .67
14.75	22.2	−.00			22.46	− .26
1	2	3	4	5	6	7

culiar conditions of the particular problem. This is true of line 0–10.75 of Fig. 1.

The distance between isoclinics is infinite for Stations 0, 1 and 2. The radii r_1, r_2 and r_3 are infinite. By Equations (8) and (9) ψ_1 and ψ_2 are 90 and 0 degrees respectively. ΔP is parallel to Q while ΔQ is parallel to P. The gradient triangle is a right angled triangle with $\Delta(P - Q)$ as the hypotenuse. This is special case a of Fig. 6. The other two sides of the triangle are:

$$\Delta P = \Delta(P - Q) \sin \theta \text{ by Equation (10)},$$
$$-\Delta Q = \Delta(P - Q) \cos \theta \text{ by Equation (11)}$$

and

$$r_2 = -\theta.$$

The ratio r_3 has a finite value at Station 3 but that value is changing very rapidly and is therefore difficult to determine. The isoclinic 86 degrees appears on the map as a line through Station 3, but since the change in slope is very slight in this region, the exact direction of the isoclinic is nearly indeterminate. The angle, therefore, which gives the direction of the isoclinic can not be accurately measured. None of the equations (8) to (11) can be solved with both r_3 and ω missing; therefore gradient values for this station are interpolated between Stations 2 and 3.5 where better measurements could be made.

The angles θ near Stations 3.5 and 4.5 are so close to 90 degrees that they are assumed to be examples of special case c of Fig. 6. By Equation (9)

$$\psi_2 = -\omega \text{ and } -\Delta Q = \frac{P - Q}{r_3}.$$

Near Station 5.5, $(P - Q)$ is a maximum and r_4 becomes infinite. This is an example of special case b of Fig. 6. $\Delta(P - Q)$ is zero and the triangle degenerates into a straight line of length $\frac{P - Q}{r_3}$. By Equations (8) and (9)

$$\psi_1 = \psi_2 = - \omega.$$

Integration calculations for the region 6 to 8.5 are very inaccurate because θ, ω and $\Delta(P - Q)$ are changing very rapidly and are therefore difficult to measure. However, θ becomes 90 degrees at about the point 8.8 where r_3 is infinite because α is a maximum. This is an example of special case a of Fig. 6 where ΔQ is zero since θ is 90 degrees. The gradient triangle degenerates into a straight line with ΔP equal and parallel to $\Delta(P - Q)$. This quantity was measurable with good accuracy. Its value having been plotted, gradient values were calculated for Stations 7 and 8.1 These less accurate values for ΔP were checked for continuity of curvature for the $\frac{dP}{ds}$ line between the better measured values at 5.5 and 8.8.

The gradient of $-Q$ changes sign at the point 8.8 where $-Q$ reaches a maximum of 4.79 as shown by column 7 of Table 2. The direction of $+P$ at this point shifts from the right to the left of the line of integration, causing the abrupt change of θ from $+90$ to -90 degrees as shown in column 6 of Table 1.

Stations 9.8, 10.3, and 10.7 are not special cases and gradients were obtained by routine measurements and calculation.

The depression in the $\Delta(P - Q)$ curve near 10.3 is introduced by the change from concave to convex equidetends in this region as shown in Fig. 1 and represents a real change in gradients in the field.

Table 2, column 6 shows that P reaches a maximum of 22.12 near Station 4. The $\frac{dP}{ds}$ value as shown by Fig. 7 reverses sign at 4.2 and the integration reduces P to 0.19 at the edge station 10.75. Since P at this point is zero, this is a very satisfactory check.

The gradient of $-Q$ remains positive from Station 0 to 8.8. Along this line $-Q$ changes from a tension of -1 at Station 0 to a compression of 4.79 at Station 8.8. The negative $-Q$ gradients beyond 8.8 reduce this value to 2.21 at Station 10.75. This checked against column 2 must show the reverse error of the last line of column 6 and is a satisfactory check.

Integration Along an Equidetend.

Integration of the line 5.5 to 14.75 of Fig. 1 is shown in Fig. 8 and recorded in Tables 3 and 4. This line follows the detend gradient from 5.5 to 13, and the $\Delta(P - Q)$ side of the gradient triangle is plotted normal to the base between these stations. At Station 13 the line of integration turns through 90 degrees to follow the equidetend 22.2 to the boundary at Station 14.75. For this section $\Delta(P - Q)$ is plotted parallel to the base. The gradient triangle for Station 13 is drawn in both positions to end integration in one direction and start it in the other.

Initial values of P and $-Q$ for line 1 of Table 4 are taken from line 5.5 of Table 2. The straight-line *gradient triangle* of Station 5.5 where θ is indeterminant takes its orientation in both Figs. 7 and 8 from the direction of the lines of integration. In the region 5.5 to 13, $\Delta(P - Q)$ was determined entirely by adjustment of a smooth curve to give the observed change in detend value, -0.8 between the two stations.

Columns 7 and 8 of Table 3 show small values for both ω and ψ_1 for the entire length of the 5.5–14.75 line. Equation (10), based on cosines of these angles was therefore reasonably independent of inaccuracies in the determination of these angles and gave reliable values of ΔP for the entire length of the line. Equation (8) was less sensitive to angle errors than Equation (9) for all stations except 14.2. At this station a θ value of -2 as compared with a ω value of 6 degrees made Equation (9) more accurate than (8). Values of ΔP and ψ_1 completed the gradient triangle for all stations except 14.2 where ψ_2 was used.

Table 4 shows in column 7 that $-Q$ is very near zero at Station 13. It is normal to the edge and hence zero at 14.75. Since the integration between these two points follows the equidetend line $P - Q$ equal to 22.2, changes in P are at all points equal to changes in Q. The net change in both P and Q between these stations is then not more than the limit of accuracy of the measurement of $P - Q$. The net area under the $\frac{dP}{ds}$ curve between these stations in Fig. 8 should be zero. The error, $-.26$ shown in the last line of columns 6 and 7, is about equal to the area between the hump and the dotted smooth section of the $\frac{dP}{ds}$ curve near Station 14. It seems probable that this hump represents mistakes in measurements near Station 14 and is a large part of the error. The appearance of this hump in the dP curve of Fig. 8 is a good example of the accuracy of graphic integration. The hump is a very prominent feature of the curve yet it represents a variation from smooth curve values which could hardly be detected by direct measurements.

The integration along the line 0–10.75 was presented because in general an integration in the direction of $\Delta(P - Q)$ is easier to plot and will give more useful information than one in any other direction. The integration along line 5.5–14.75 was presented to show that a line of integration can be continuous, although its direction with reference to the equidetend and isoclinic pattern may change.

CONCLUSION.

Stress, a condition in a body resulting from its deformation, is a second-order tensor which is defined at every point in the body by three principal stresses and three orthogonal orientations which direct them. A principal stress is evaluated by the force it maintains across unit normal area. Neither a principal stress nor the force which evaluates it can be divided into real components taken in some other direction. Either of them can be divided into any desired combination of imaginary components which need not be differentiated from real components for most mathematical operations. Only a physical cause can give reality to force or stress components. Shear is as imaginary force components parallel to a given plane.

Stress gradients in a static stress field depend on the curvatures of the stress-axis lines and the differences of the principal stresses. The principal stresses themselves do not enter the equations of variation independently. The new words, *detend* $(P - Q)$ for plane stress and *double detend* $(P - Q)$, $(P - R)$ for three-dimensional stress, are introduced to identify these stress-gradient-determining factors. A *detend* is a property of a principal stress and has the orientation of its stress. It is invariant for any point is a stress field. It differs from shear which is a property of a plane, is variable for all points of the field, and is always zero in the directions of the principal stresses.

Stress gradients in any plane, static stress field can be analyzed graphically by the use of a *stress-gradient triangle* whose sides represent the maximum gradients $\Delta(P - Q)$, ΔP and $+\Delta Q$. The *stress-gradient* triangle for any point in a stress field is determined by five quantities, $(P - Q)$, θ, ω, r_3, and r_4 which can be measured directly in any photoelastic field. When all five quantities can be measured the determination of the *gradient triangle* is doubly redundant.

Integrated changes of $P - Q$, P and $+Q$ along any line over a plane static stress field are proportional to the area swept out by the normal projections of the sides of the stress *gradient triangle*, rotated counterclockwise through 90 degrees, as it moves along the line of integration. These areas can be summed graphically by plotting the line of integration straight, independently of its curvature or angular turns over the stress field.

The convenient check given by the redundance in the determination of the stress gradient triangle may be the chief advantage of its use.

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THE PENNSYLVANIA ASSOCIATIONS OF JOSEPH PRIESTLEY.

BY

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Joseph Priestley was born at Fieldhead, Yorkshire, England, on March 13 (Old Style), 1733. He was educated at the Daventry Academy, a seminary for the training of Dissenting clergymen, which he attended from 1752 to 1755. He then served as pastor at Needham Market (1755–1758) and Nantwich (1758–1761), as tutor in the Warrington Academy (1761–1767), and as pastor of the Mill Hill Chapel, Leeds (1767–1773). Then followed a period (1773–1780) spent in the service of William Fitzmaurice, second Earl of Shelburne, afterwards first Marquis of Lansdowne. In 1780 he became pastor of the New Meeting, Birmingham, and held that position until 1791. He was ordained in 1762. Originally a Calvinist, he finally became a Unitarian. In London, he met Benjamin Franklin, then (1764–1775) agent there for the Province of Pennsylvania; Franklin aided and encouraged Priestley in the composition of the latter's treatise "The History and Present State of Electricity," which appeared in five editions during its author's lifetime, the first edition in 1767. Priestley was awarded the degree of Doctor of Laws by the University of Edinburgh in 1765, and was elected a Fellow of the Royal Society in 1766, a foreign associate of the Academy of Sciences of France in 1772, a foreign honorary member of the American Academy of Arts and Sciences in 1782,[1] and a member of the American Philosophical Society in 1785. Priestley became the foremost pneumatic chemist of his period, which was the period of pneumatic chemistry. His first research in this field was on "Fixed Air" or carbon dioxide, its solution in water, and the use of that solution as an antiscorbutic. This research was published in 1772, and brought to Priestley an award of the Copley Medal in 1773 by the Royal Society. One result of this research was soda water. His researches were published as "Experiments and Observations on Different Kinds of Air" in six volumes between 1775 and 1786. He invented a pneumatic trough for the collection of gases. When mercury was used in this apparatus, the discovery, isolation, and study of water-soluble gases, such as ammonia and hydrogen chloride, were made possible. Priestley discovered and described the following *gases*: Nitric oxide, nitrous oxide, hydrogen chloride, ammonia, sulfur dioxide, silicon fluoride, and hydrogen sulfide; he also discovered carbon monoxide and nitrogen, but failed to recognize their true nature. His greatest discovery was oxygen ("dephlogisticated air"); he apparently obtained oxygen in 1771 by heating nitre in a gun-barrel; however the actual discovery was made on August 1, 1774, when the new gas was liberated from mercuric oxide by placing the latter in the focus of a burning lens of considerable force. Other discoveries of Priestley were nitrosulfuric acid (the lead chamber crystals of the sulfuric acid plant) and the restoration of vitiated air by green vegetation in the presence of sunlight.

Priestley sympathized with the French Revolution. Consequently, during the Birmingham riots on July 14, 1791, not only the New Meeting and the Old Meeting, but also Priestley's residence were destroyed by the mob. By court action, Priestley was subsequently awarded two thousand five hundred two pounds, eighteen shillings damages, which were paid by a rate or tax on the hundred or political unit concerned.

Thereafter Priestley remained in London until 1794. On April 7, 1794, he sailed from Gravesend, and arrived in New York on June 4, and in Philadelphia on June 19. In mid-July of that year he removed to Northumberland, Pennsylvania, where he resided until his death on February 6, 1804. He remained a British subject. His remains were interred in the Quaker Green at Northumberland on February 9, 1804. Linn,[2] writing in 1873, stated that the remains "within a few years since" were removed to the beautiful cemetery on the outer edge of the town, Riverview Cemetery. The legend on his tombstone is:

> To
> The memory of the
> Rev'd. Dr. Joseph Priestley
> who departed this life
> on the 6th Feb'y 1804
> Anno Etatis LXXI
>
> Return unto thy rest, oh my soul for the
> Lord hath dealt bountifully with thee.
> I will lay me down in peace and sleep till
> I awake in the morning of the resurrection.

Priestley's "Memoirs" [3] were published in 1806; among his biographers have been Anne Holt [4] and T. E. Thorpe [5,6]. Edgar F. Smith [7,8] has paid tribute to his work in America; and Schuster and Shipley [9] have ranked him as one of the great men in the history of British science. Fulton and Peters [10,11] have published bibliographies of Priestley.

The association of Priestley with Pennsylvania may be considered under these headings: His home at Northumberland, his apparatus and its fate, his connection with the American Philosophical Society, the University of Pennsylvania, the Unitarian Church, the Northumberland Academy and its subsequent history, and lastly, the centennial of the discovery of oxygen and the organization of the American Chemical Society.

PRIESTLEY'S HOME.

Priestley erected his home on Northway in Northumberland on a parcel of real estate, containing two acres, forty-two perches, which he purchased from the heirs of Reuben Haines, brewer of Philadelphia, to wit, Caspar Wistar Haines, Reuben Haines, Josiah Haines, and Catharine Haines, by indenture made in October, 1795, for one hundred and four pounds, twelve shillings. The construction of the house from kiln-dried lumber required two and a half years.

On May 13, 1815, the younger Joseph Priestley and Elizabeth, his wife, then in England, by their attorneys, Thomas Cooper and John Vaughan, sold the property to Seth Chapman, then President Judge of the Eighth Judicial District of Pennsylvania, then composed of the counties of Northumberland, Luzerne, and Lycoming; Chapman [12] held that office from 1811 to 1833. He paid six thousand, two hundred and fifty dollars for the property.[13]

The title passed to Robert George Scott, whose family resided in the house from 1888 to 1919.

In 1919, the title was conveyed by Robert F. Scott and Flo J. Scott, his wife, James W. Scott and Sarah Jane Scott, his wife, and Robert S. Whitmer to George Gilbert Pond, who became professor of chemistry in The Pennsylvania State College in 1888 and dean of its school of natural science in 1896, and held both positions until his death on May 20, 1920.

The estate of Dr. Pond conveyed the title to The Pennsylvania State College.

At its Session of 1943, the General Assembly of Pennsylvania passed an Act (Senate Bill 350) which authorized the acquisition of this property by the Commonwealth of Pennsylvania by gift from The Pennsylvania State College as a permanent museum and historic shrine. This Act also made an appropriation for maintenance and development of the property. On June 1, 1943, by Veto No. 15, the Governor vetoed the Act.

> "This item was not included in the budget, and the condition of State revenue does not warrant the taking over at this time of additional historical sites to be maintained by the Commonwealth.
>
> "For this reason the bill is not approved."[14]

Priestley's home contained his library. The books, which constituted this library, were sold in 1816 by Thomas Dobson of Philadelphia. One wing or "L" of Priestley's home was his laboratory.

In his letters to John Vaughan at Philadelphia, Priestley [15] at Northumberland several times mentions supplies for his laboratory. On December 21, 1794, he apparently concluded a now somewhat mutilated letter: "Whenever I begin to make experiments I shall want a quantity of *oil of Vitriol.* If you can send me *two gallons* at earliest opportunity you will oblige me much." On July 28, 1796, Priestley wrote that he had received a spirit level and three small lenses for a microscope. The level, "not answering the purpose," was returned. The lenses were worn, and were likewise returned. Priestley then expresses his disappointment concerning apparatus of American manufacture: "I find myself exceedingly disappointed, as I had told my friends, and really believed, that I could now be furnished with philosophical instruments in America as well as in England. This I now fear is at a great distance."

On March 21, 1799, Priestley[15] wrote concerning the damage to glass apparatus during passage through the customs and as a result of improper repacking: "I sent to my son as particular account as I could give of the *package of glass*, and did not doubt but that he would have given it to you. It contained nothing but glasses of very various forms for philosophical purposes, and I have been told that philosophical instruments do not pay any duty. The value I believe to be in England *five* or six *guineas*, but being a present to me, I do not know exactly. I have another box of philosophical instruments coming on the *Neptune*, about which I wish you to speak to the proper officers. If they be unpacked at the Customs house, they may almost as well remain there, as it will be impossible to replace them without great damage, for which I suppose nobody would be answerable. I was several hours in unpacking the glass that came last, and I could not have put half the things into the same box again. Many were broken in consequence of too close packing. My direction to my correspondents in England is always to acquaint you with the particulars of any package for me; but those who occasionally send me presents cannot always be apprized of this. There should be some better legal provision in this case. Philosophical instruments, that are not made here, are much wanted in this country, and if your rules were strictly observed, it would amount to a prohibition of them. Without the packages being opened at the customs house, a very great proportion of the articles sent hither are broken or much damaged. And broken glass is of no use. If the packages must absolutely be opened, I should never order anything more." The package in question contained retorts, a phial with ground, stoppered tubes, jars, evaporating dishes, tubes, and, apparently, some highly specialized apparatus, "things that have no names and require drawings to give an idea of them."

On April 11, 1799, Priestley[15] again wrote concerning his laboratory work: "I wish also to continue my *experiments*, which are necessarily attended with some expense. Can I trouble you with procuring for me against the next journey of the waggon, *half an ounce of pure gold*, and *an ounce of pure silver* from your assay office? If you can conveniently advance thus much, I shall be obliged to you. If not, I will send the money by the next waggon. . . . I shall not wholly despair of your joining our little quiet society here, and seeing you now and then in my laboratory."

He published the results of certain of his researches in the *Transactions of the American Philosophical Society* as noted below. He also published at Northumberland in 1800 a book of 106 pages entitled "The Doctrine of Phlogiston Established and that of the Composition of Water Refuted"; a second edition with additions appeared at Northumberland in 1803, and contained 143 pages.

Priestley owned other land in Pennsylvania.[16] He purchased 337 acres in the Loyalsock district, near Williamsport, from Dr. Benjamin Rush for five shillings on February 10, 1794.[17]

The descendants of Priestley, who resided in Northumberland, lived in a house on King Street now the Priestley-Forsyth Public Library.

A fire-proof museum to house Priestleyana was erected on the lawn of Priestley's home, and was dedicated on September 5, 1926.[18]

Priestley had no permanent home in Philadelphia. The statement by Lippincott [19] that Priestley dwelt on the north side of Market Street between Sixth and Seventh Streets apparently refers to his abode while visiting Philadelphia.

PRIESTLEY'S APPARATUS.

Several institutions possess apparatus from Priestley's laboratory at Northumberland.[18]

The Franklin Institute of the State of Pennsylvania for the Promotion of the Mechanic Arts owns Priestley's air pump.[20] This pump was procured from the Priestley family by their friend Charles Sellers (1806–1898) for use in lectures at Cincinnati; it passed, about 1849, to his younger brother, George Escol Sellers (1808–1899), who sent it to another brother, Coleman Sellers (1827–1907), in 1897. The sons of Coleman Sellers, Coleman Sellers, Jr. and Horace Wells Sellers, presented the pump to the Franklin Institute on October 20, 1920.

Dickinson College [21,22] at Carlisle, Pennsylvania has Priestley's burning lens, his air gun, telescopes, and other apparatus, which were procured for it by Thomas Cooper,[23] its professor of chemistry, from 1811 to 1815, and a friend and associate of Priestley. Cooper (1759–1839) was born in England, educated at Oxford, studied both law and medicine, and was a member of the National Assembly of France during the Reign of Terror. He was insulted, during a sitting of the Assembly, by Robespierre, whom he then denounced as a knave and a rascal. He then had to flee France to escape the guillotine. In Pennsylvania, he was president judge of the Fourth Judicial District (1804–1811). At the University of Pennsylvania, he was professor of mineralogy and chemistry (1816–1821). From 1820 to 1834, he was professor of chemistry in and president of the University of South Carolina.

Certain apparatus of Joseph Priestley was deposited in the Smithsonian Institution at Washington, D. C, by the heirs of Dr. Joseph Priestley, 4th, in 1883.

Annie Priestley, the great-great-granddaughter of Joseph Priestley, and the last of the Priestley name to reside in Northumberland, died in 1922. She and her sisters presented to the late Edgar Fahs Smith, thirteenth Provost of the University of Pennsylvania, the relics of Joseph Priestley described by Smith in 1923 in his brochure, "Priestleyana." [24] Smith also possessed Priestley's balance. This collection was returned to the museum which was erected on the estate of Joseph Priestley.

PRIESTLEY AND THE UNIVERSITY OF PENNSYLVANIA.

The election of Priestley to the professorship of chemistry in the University of Pennsylvania, and his declination of that election are recorded in Volume 5 of the manuscript "Minutes of the Trustees of the University of Pennsylvania," [25] from which these excerpts have been made.

On Tuesday, November 11, 1794: "A letter from Doctor Barton, requesting his name as a candidate for the professorship of Chymistry may not interfere with that of Doctor Joseph Priestly, was received and read.

"The board according to order proceeded to the election of a Professor of Chymistry in the room of Doctor John Carson, deceased: when the ballots being taken and counted it appeared that Doctor Joseph Priestley, was unanimously elected."

On Tuesday, March 3, 1795, the Board met, and Mr. Chief Justice McKean was appointed Chairman. "Mr. Chief Justice informed the Board, that Doctor Joseph Priestley, had declined the Professorship of Chymistry, to which he was elected in this Institution, the 11th November last." Thomas McKean was then the Chief Justice of the Supreme Court of Pennsylvania.

On Tuesday, June 2, 1795: "A letter from Doctor Benj. Rush, was read, recommending Dr. James Woodhouse for the professorship of Chymistry: A petition from several of the students in Medicine in the University, upon the same subject; and a letter from Dr. James Woodhouse, offering himself a candidate for that Professorship were also read.

"The Board nominated several gentlemen out of whom to make a choice of a Professor of Chymistry, in the room of Doctor Joseph Priestley, who has declined the appointment."

On Tuesday, July 7, 1795: "The Board proceeded to the election of a Professor of Chymistry, in the room of Dr. Joseph Priestley, who has declined serving, and the Ballots being taken & counted, it appeared that Doctor James Woodhouse was duly elected."

The following excerpts from Priestley's "Memoirs" [3] explain his declination of the professorship of chemistry, and also his relationship to the office of provost of the University:

> "The following reasons among others induced him to prefer a country to a city life. He thought that if he undertook the duties of a professor, he should not be so much at liberty to follow his favourite pursuits as he could wish, and that the expence of living at Philadelphia or New York would counterbalance the advantages resulting from his salary; . . . ; but what had greater weight with him than anything else was that my mother, who had been harrassed in her mind ever since the riots at Birmingham, thought that by living in the country, at a distance from the cities, she should be more likely to obtain that quiet of which she stood so much in need.
>
> ……
>
> "I shall in this place state, though I shall anticipate, in so doing, that in the year 1803 a vacancy occurred in the University of Pennsylvania, by the death of Dr. Euen, Principal of that institution. It was intimated to my father by many of the Trustees, that in case he would accept of the appointment, there was little doubt of his obtaining it; Mr. M'Kean, the present governor of the State of Pennsylvania, being among others particularly anxious that he should accept of it. In addition to the reasons that had induced him to decline the offer of the Professorship of Chemistry were to be added the weak state of his health, which would have made the idea of his having any serious engagement to fulfil, very irksome to him; he accordingly declined it.
>
> "He had frequent intimations of other proposals of a similar nature that would have been made to him, had it not become generally known, that he could not acceed to them from their being inconsistent with the plan of life he had laid down for himself."

"Dr. Euen, Principal" was the Rev. John Ewing, Doctor of Divinity of the University of Edinburgh (1773), Provost of the University of the State of Pennsylvania (1779–1791) and Provost of the University of Pennsylvania from 1791 until his death on December 7, 1802. He had been incapacitated since January of that year. "Mr. M'Kean"was The Hon. Thomas McKean, Governor of the Commonwealth of Pennsylvania (1799–1808), the "Mr. Chief Justice McKean" of the minutes cited above.

This tender of the office of Provost to Priestley apparently was the act of individual Trustees, for Volume 5 of the manuscript "Minutes of the Trustees of the University of Pennsylvania" makes no mention of either nomination or election of Priestley to that office in the period between the quasi-retirement of Ewing and the death of Priestley.

PRIESTLEY AND THE AMERICAN PHILOSPHICAL SOCIETY.

The relationship of Joseph Priestley to the early history of the venerable American Philosphical Society has been discussed by Bronk.[26] It is also revealed by study of the volume of early proceedings [27] of that Society from 1744 to 1838, where the name of Priestley appears over fifty times. On January 22, 1785, he was elected a member and his certificate of membership in the Society is now among the treasures in its library. Upon his arrival in Philadelphia, the Society adopted a congratulatory address on June 20, 1794. His reply is noted in the minutes for July 18, 1794. He attended meetings of the Society, in its Hall in the State House (Independence Hall) Square which it has occupied continuously since 1789, several times in each of the years 1796, 1797, 1801, and 1803. On February 3, 1797, he was elected orator for the meeting on the last Saturday of December, but declined the election on February 10, 1797. In March 1803, the Society tendered Priestley a dinner. On February 17, 1804, the Society ordered a eulogium on Priestley, and, on February 24, 1804, elected Dr. Barton as the eulogist. The eulogy was delivered in the First Presbyterian Church on January 3, 1805; the ceremony was preceded by a procession from the Hall of the Society to the Church. On January 18, 1805, this eulogium was ordered published in the *Transactions* of the Society. But on November 6, 1812, Barton requested and, on November 20, 1812, was granted permission to withdraw the eulogium on Priestley to be enlarged and published separately. Unfortunately, this separate publication apparently never occurred. On September 16, 1791, a profile in plaster of paris of Priestley "particularly valuable for the Resemblance" was presented to the Society. Other gifts were: A medallion in plaster of Priestley on April 1, 1803; a profile of Priestley in black leather on April 6, 1804; and a portrait of Priestley "long in the possession of our late president" (Dr. Caspar Wistar) on April 3, 1818. Priestley donated copies of certain of his publications to the Society, and submitted to it certain of his papers. In fact, he contributed eleven papers to the three volumes (4, 5, and 6) of its *Transactions* which appeared, respectively, in 1799, 1802, and 1809.

To volume 4 he contributed "Experiments and Observations relating to the Analysis of Atmospherical air" (pages 1-11, read February 5, 1796), "Further Experiments relating to the Generation of Air from Water" (pages 11-20, read February 19, 1796), and "An Appendix to the two Articles in this Volume (p. 1)" (pages 382–386, read November 23, 1798 as a letter to B. S. Barton, M.D.).

The first six papers in volume 5 were from his pen: "Experiments on the Transmission of Acids, and other Liquors, in the form of Vapour, over Several Substances in a hot earthen Tube" (pages 1–13, read December 20, 1799), "Experiments relating to the Change of Place in different kinds of Air through several interposing Substances" (pages 14–20), "Experiments relating to the Absorption of Air by Water" (pages 21–27), "Miscellaneous Experiments relating to the Doctrine of Phlogiston" (pages 28–35), "Experiments on the Production of Air by the Freezing of Water" (pages 36–41, read April 18, 1800) and "Experiments on Air exposed to Heat in metallic Tubes" (pages 42–50, read August 15, 1800).

Volume 6 contains two papers by Priestley: "Observations and Experiments relating to equivocal, or spontaneous, Generation" (pages 119–129, read November 18, 1803), and "Observations on the Discovery of Nitre, in common Salt, which had been frequently mixed with Snow" (pages 129–132, read December 2, 1803, as a letter to Dr. Wistar).

The *Transactions* also contain (volume 4, pages 181–215) a lengthy letter, read May 20, 1796, from Benjamin Smith Barton to Priestley, entitled "Observations and Conjectures concerning certain Articles which were taken out of an ancient Tumulus, or Grave, at Cincinnati, in the County of Hamilton, and Territory of the United States, North-West of the River Ohio."

PRIESTLEY AND THE UNITARIAN CHURCH.

Joseph Priestley was, by profession, a Unitarian clergyman. In 1796, by invitation, he delivered in Philadelphia a course of lectures on "The Evidence of Revealed Religion." Asked to form a church, and become its minister, Priestley declined the office, but gave his assistance in the organization of a church and became one of its lay members. The organization of this church,[28] known originally as "The First Society of Unitarian Christians in Philadelphia," now as "The First Unitarian Church of Philadelphia," is commemorated by one of two tablets which have been erected on the south wall of 62 North Fourth Street, in the driveway immediately south of that building. The legend on one tablet is:

In a Building
of the
University of Pennsylvania
which stood near this spot
"The First Society of Unitarian
Christians in Philadelphia,"
being the first church in America
to adopt the Unitarian name,
was organized 12 June 1796,
under the influence of
Joseph Priestley, LL. D., F. R. S.
celebrated theologian
and philosopher
discoverer of oxygen and
founder of modern chemistry
inflexible defender of human rights
this tablet is erected by
the Unitarians of Philadelphia
4 October 1908

The legend on the other tablet is:

On this site stood the "New
Building" erected in 1740
for George Whitefield
and for a Charity School
subsequently until 1802
it was used by the School,
Academy, College and
University of Pennsylvania
successively—This tablet
was placed here by the
Class of 1889 College on the
twentieth anniversary of
their graduation June 1909

The present building of this Church, at the northwest corner of Chestnut and Van Pelt Streets, contains a Priestley Memorial Chapel, a large tablet and bust of Priestley, erected as a tribute to his memory by the Unitarians of America in 1889, and other Priestleyana.

Priestley is also credited with founding the Unitarian Church at Northumberland; its present edifice was erected in 1834. The church was remodeled and rededicated in 1910 as "The Joseph Priestley Memorial" through the interest of the American Unitarian Association.[29]

The first signature on the "paper" under which the "Society" was first organized in Philadelphia on June 12, 1796 is that of Joseph Gales,[30] who was a friend of Joseph Priestley, a journalist, publisher of *Gales's Independent Gazetteer* in Philadelphia, founder of the *Register* in Raleigh, North Carolina, and state printer of that state.

The memory of Joseph Priestley is perpetuated by "The Joseph Priestley House" which was incorporated by decree of Court of Common Pleas No. 5 of Philadelphia County on October 27, 1931.[31] It is a home for aged persons at 224 West Tulpehocken Street, Philadelphia. All the members of its board of trustees are Unitarians. Under its by-laws, while preference for admission is given to Unitarians, other persons are eligible.

(*Continued in next issue*)

THE FRANKLIN INSTITUTE.

MEDAL DAY PROCEEDINGS.

The annual reception, dinner and presentation of awards, known as "Medal Day," took place at The Franklin Institute on Wednesday, April 16, 1947. The reception began at 6:30 P.M. and dinner was served at 7:15 P.M. Mr. Richard T. Nalle, President of the Institute, presided at the dinner and at the subsequent awards.

RECEPTION.

The reception preceding the dinner was given by the Hostess Committee of The Franklin Institute. Approximately four hundred and five persons attended the reception which was held in rooms adjoining the Chemistry Section. Dinner was served in Franklin Hall.

Immediately before dinner was served, Captain Guy Marriner played the National Anthem.

At the close of the dinner, Mr. Nalle asked the guests to pay tribute to Benjamin Franklin by drinking, in wine, a toast to his memory.

STATED MONTHLY MEETING.

The President called upon the Sponsors for the 1947 Recipients of awards to take their places. He then announced that this was a regular stated monthly meeting, that the minutes of the February meeting had been published in the March issue of the JOURNAL and, if there were no objections, they would be declared approved as printed. As there was no dissent, the minutes were declared approved as printed.

MR. NALLE (continued): "Speaking for the members of the Board of Managers and the officers of The Franklin Institute, it is a great pleasure and satisfaction to have you here tonight at our Medal Day meeting. (The Institute has awarded medals since its founding for outstanding achievement in various lines of endeavor.)

"We are assembled to honor 19 gentlemen for outstanding accomplishment in their chosen scientific fields.

"At this meeting, each of the 19 gentlemen will receive either The Franklin Institute certificate or medal.

"The Hostess Committee, under the chairmanship of Mrs. John White Geary, is responsible for the reception before dinner, and I give the thanks of all of us to Mrs. Geary and her Committee for the very pleasant occasion which they provided.

"It is our custom to invite to this annual meeting Recipients of awards of previous years. There are 26 such with us tonight and we welcome them. I shall read their names and award received and ask that each one rise as his name is called and remain standing until the roll is completed. To save time, will you kindly withhold your applause until all are standing?"

FORMER MEDALISTS ATTENDING MEDAL DAY—1947.

Year of Award	Medalist	Medal
1904	Clamer, Dr. G. H.	Cresson
1916	Dorr, Mr. John V. N.	Scott
1920	Kothny, Mr. G. L.	Longstreth
1921	Adams, Dr. L. H.	Longstreth
1923	Hull, Dr. Albert W.	Potts
1923	Parks, Mr. Harry S.	Longstreth
1924	McBride, Mr. Thomas C.	Longstreth
1924	Fletcher, Dr. Harvey	Levy
1930	Chrisman, Mr. Charles S.	Wetherill
1932	Harrison, Mr. Thomas R.	Certificate of Merit
1932	Wenner, Dr. Frank	Wetherill
1935	Shrader, Dr. James E.	Wetherill
1936	Hall, Mr. Peter P. G.	Longstreth
1937	Eksergian, Dr. Rupen	Henderson
1945	Eksergian, Dr. Rupen	Levy
1938	Russell, Mr. N. F. S.	Longstreth
1938	Grondahl, Dr. Lars O.	Potts
1918	Creighton, Dr. H. J.	Longstreth
1939	Creighton, Dr. H. J.	Potts
1939	Wilford, Mr. E. Burke	Certificate of Merit
1940	Slayter, Mr. Games	Longstreth
1941	Black, Mr. Harold S.	Wetherill
1941	Wilson, Mr. Benjamin	Longstreth
1944	Ennis, Mr. Joseph B.	Henderson
1944	Rentschler, Dr. Harvey C.	Brown
1944	Stokes, Mr. J. Stodgell	Longstreth
1946	Barnes, Maj. Gen. G. M.	Cresson
1946	Southworth, Mr. George C.	Levy

MR. NALLE: "The regular order of business calls for a report from our Secretary, Dr. Henry Butler Allen, who was also appointed the Executive Vice President of the Institute several months ago. Dr. Allen will now present his report to which he has given the title, 'The Institute in 1947.' Dr. Allen."

REPORT OF THE SECRETARY.

DR. ALLEN: "Mr. President, Honored Guests, Friends of the Institute.

"Again, this year, I am going to tell you briefly about what has been going on here since last April. This time in outlining our activities I am going to bear in mind a constantly recurring query, 'Just what does The Franklin Institute do?; we know you are called an educational institution, you give medals, you have a Museum and Planetarium; but do you have classes?,' etc., etc. I don't wonder at such queries, because on the face of it, we may seem to have an out of the ordinary lot of activities. But actually behind it all there is a pretty definite objective and plan; and we have been consistent since we started in the year 1824.

"We bear the corporate name—The Franklin Institute of the State of Pennsylvania for the Promotion of the Mechanic Arts. We were founded 34 years after our 'patron saint's' death, to fill an existing gap toward carrying out one of his life-long interests—to promote American manufacture and help the young mechanic and engineer get ahead.

"So just what does the Institute do?—and how are we doing it? Perhaps the clearest answer can be given by listing our main fields of activity:

"*Our Lectures* consist of technical, semi-technical, and popular talks on recent developments in science; and we are fortunate to be able to get the men and women who are responsible for these advances 'in person' to give these talks.

"The technical and patent *Library* continues to serve an ever increasing number of manufacturers, research workers and students. It is, of course, a great asset to our own research laboratories.

"Our *Medal Awards* speak for themselves this evening. Again during the past year I have had the great satisfaction of having more than one past medalist tell me that his medal from us in earlier days had given him encouragement to go ahead to greater things.

"The *Museum* is now beginning to recover from the war years—as rapidly as we can get the financial support. (I can think of nothing more satisfying to the family of a departed member of the Institute than the endownment of one of our science exhibits.) And only last week we had a letter from a Washington patent lawyer about a technical matter, who closed his letter by saying, 'I cannot close without telling you how much Mrs. Patch and our two sons, as well as myself, have enjoyed being permitted to visit the Institute, and the Planetarium. We have made several trips to Philadelphia for the sole purpose of visiting the Institute; and, I find that now very often both of my sons, presently attending college and studying Civil Engineering and Chemical Engineering, respectively, mention something they saw in the Institute, and which now helps them in their college work and in a better understanding of the principles involved. For my own part, I have been particularly helped by the clear and simple manner in which your exhibits demonstrate principles which are sometimes treated in such abstract manner by the books and instructors.'—Here is a graphic incident that occurred a month ago to show how youth is catching on to the new science. We had a little demonstration on atomic energy. Our demonstrator, using words of one syllable, referred to certain gaseous molecules as being broken up by cosmic rays. After the demonstration, an eight-year-old boy came up and said to the lecturer,' You meant that the molecules were ionized, didn't you?'

"I could talk for an hour about our *Research*, both our newer industrial research and development laboratories, and those for so-called pure science research—our Bartol Foundation and the Biochemical Research Foundation. But I will only say that the former are giving real service to national preparedness, making a definite start in helping industry, and have grown during the past year to 270 workers. We cordially invite any who are interested to visit us and see how we might help solve your problems. And right now we can show you our newest piece of equipment, an electron microscope—developed, by the way, by one of our medalists of tonight, Dr. Zworykin, and who, two weeks from tonight, will demonstrate color television for the first time outside his laboratory.

"Lastly, our *National Franklin* activity is receiving a steadily increasing demand for our little pamphlets about the teachings of our patron saint. It is most encouraging to see the sources of these requests and the use made of the information. People are really using Franklin to help their thinking about how to make their lives more useful."

MR. NALLE: "Recommendations for the awards given by The Franklin Institute are made by the Institute's Committee on Science and the Arts to the Board of Managers for their approval. The Committee on Science and the Arts is composed of some 65 members with a very broad experience in many different fields of science. They meet regularly once a month and their sub-committees as frequently as necessary. They are constantly on the lookout for the noteworthy accomplishments in scientific work. Recommendations for possible awards are studied, discussed and analyzed very thoroughly by the sub-committees and finally are brought before the entire committee for review.

"I assure you that it is not a haphazard procedure. To attend their meetings is an intellectual treat, and to be recommended for an award an outstanding achievement. The Board of Managers considers the Committee on Science and the Arts one of the Institute's most valuable assets.

"We will now proceed to the presentation of the awards."

Mr. Nalle, then called upon the sponsors to present their candidates for the awards.

PRESENTATION OF THE CERTIFICATES OF MERIT.

(To Joseph Razek, Havertown, Pennsylvania, and Peter J. Mulder, Letterkenny Ordnance Depot, Chambersburg, Pennsylvaina.)

MR. NALLE: "The Chair recognizes Mr. C. H. Masland, 2nd."

MR. MASLAND: "Mr. President, I introduce Joseph Razek, Consulting Physicist, of Havertown, Pennsylvania, and Peter J. Mulder, of Letterkenny Ordnance Depot, Chambersburg, Pennsylvania.

"From time immemorial the question 'What color is it?' has been the basis of many arguments. Seller and buyer, dyer and weaver cannot agree. Two observers of the same biological specimen cannot agree on its color. Nor can observers make a quantitative statement concerning a color for future reference.

"Our Medalists have invented an instrument which analyzes and records intensity of light in its different component wave lengths whether the light is reflected from an opaque surface, transmitted through a translucent substance or comes directly from the light source to the instrument. This is done automatically and objectively without reference to the eyes or brain of the observer. And what is more important in the case of rapidly changing colors, the color of a sample can be analyzed within a few seconds of time.

"I present Joseph Razek and Peter J. Mulder to each receive a Certificate of Merit awarded to them upon citation reading:

> 'In consideration of their perfection of a means for making rapid, automatic color comparisons and analyses of opaque and translucent materials, and of light sources.'"

MR. NALLE: "Dr. Razek, by virtue of the power vested in me as President of The Franklin Institute of the State of Pennsylvania, I present to you this Certificate of Merit and the report which accompanies it.

"Dr. Mulder, likewise, by virtue of my office, I present to you the Certificate of Merit and the Report which accompanies it."

PRESENTATION OF THE LONGSTRETH MEDALS.

(To Samuel Berman, Waugh Laboratories, New York, N. Y., and Harold John Warren Fay, Submarine Signal Company, Boston, Massachusetts.)

MR. NALLE: "The next Sponsor, Dr. G. S. Crampton."

Dr. Crampton: "Mr. President, I introduce Samuel Berman, Associate Director, Waugh Laboratories, New York City.

"It must be stimulating, indeed, to a layman to feel that through his ingenuity he has assisted in guiding the operator's hand in two of the most delicate fields of surgery, namely those of the eye and the brain.

"The X-ray, by general orientation of a foreign substance in the body, paves the way for a more exact technique of indicating the path for the scalpel.

"Our Medalist, while not the first to use electronic methods in this field, has devised a sensitive electrical detector and accessory equipment for the location of foreign metallic substances in the body, which greatly assist in their removal.

"The device acts as an electronic probe which indicates by a sound of increasing pitch its approach to the substance to be located, and also by indication on a dial on a connected control unit.

"Even quite small particles can be thus detected and approximately located.

"I present Samuel Berman to receive an Edward Longstreth Medal, upon citation reading:

> "'In consideration of the development of a Metal Locator for use in surgery which device has greatly facilitated the location and removal of foreign metallic bodies imbedded in the tissues.'"

MR. NALLE: "Mr. Berman, by virtue of the power vested in me as President of The Franklin Institute of the State of Pennsylvania, I present you with the Edward Longstreth Medal, and the Certificate and copy of the Report which accompanies it.

"Professor C. M. Gay."

PROFESSOR GAY: "Mr. President, I introduce Harold John W. Fay, President of Submarine Signal Company, of Boston.

"Mariners must, of course, know the depth of waters they navigate and the location of reefs and shoals.

"Until fairly recently, soundings were made by a weight, usually of lead, attached to a line and dropped into the water until the weight strikes the bottom; when the length of line paid out gives the depth.

"This was laborious and inaccurate, as high peaks of reefs or shoals which vessels could strike were too frequently missed and escaped charting; unless otherwise disclosed by a sweep wire.

"About twenty years ago experiments were begun to send a sound signal from the bottom of a vessel to the water bottom, from where it was echoed back, the elapsed time between signal and receipt of the echo making it possible to determine the depth of the water.

"Satisfactory operation was not attained in World War I, however, and therefore our Medalist devoted himself perseveringly to improvements.

"By the time of World War II he had developed dependable devices, both audible and supersonic, of greatest service to mariners and to our Naval vessels in unfamiliar or poorly charted and uncharted waters; and for rapid depth surveys for accurate charting.

"And, of perhaps even greater importance, he produced an echo-listening device for detection of submarines.

"Our Navy was thus provided with superior navigation safety and submarine detection devices.

"I present Harold John W. Fay, to receive an Edward Longstreth Medal, upon citation reading:

> "'In recognition of his foresight, devotion, scientific acumen and practical outlook in the initiation and development of the art of underwater signalling and listening.'"

MR. NALLE: "Mr. Fay, by virtue of my office, I present to you the Edward Longstreth Medal and the Certificate and Report which accompany it."

PRESENTATION OF POTTS MEDALS.

(To Robert Harrington Kent, Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland, and Vladimir Kosma Zworykin, Radio Corporation of America, Princeton, New Jersey.)

MR. NALLE: "I call upon Colonel H. B. Ely."

COLONEL ELY: "Mr. President, I introduce Robert Harrington Kent, Associate Director of the Ballistics Research Laboratory at the Aberdeen Proving Ground.

"Our Medalist has devoted most of his active life to the science of ballistics, that branch of gunnery dealing with the motion and impact of projectiles.

"He has notably advanced this science, and also has developed the supersonic wind tunnel at the Proving Grounds to a state of usefulness in scientific research.

"I present Robert Harrington Kent to receive a Howard N. Potts Medal awarded to him upon citation summarizing his achievements and reading as follows:

> "'In consideration of his outstanding contributions to the science of ballistics and particularly to his practical adaptation of scientific methods to the improvement of ordnance; for his development to a state of usefulness to science of the Aberdeen supersonic wind tunnel; and for his contributions in ballistics and aerodynamics.'"

MR. NALLE: "Mr. Kent, by virtue of my office, I present to you the Howard Potts Medal and Certificate and Report which accompany it."

MR. NALLE: "Dr. C. B. Bazzoni, will you take the floor?"

DR. BAZZONI: "Mr. President, I introduce Vladimir K. Zworykin, Director of Electronic Research, the Radio Corporation of America.

"Epochs in civilization are frequently introduced by great inventions. To the telegraph and telephone, each of which in its turn exerted a profound influence on human progress, we have in very recent years added television. Television in affording man means of observing visually events at a distance simultaneously with their occurrence promises to play an important part in the instruction and edification of the public in peace time, just as it has already demonstrated its versatility in numerous applications in time of war.

"Modern television involves highly complex arts which rest on the researches of a large number of scientific workers, amongst whom one man stands out so sharply that he can justly be called the Father of Present Day Television. We wish today to recognize the manifold achievements of this man in this field—achievements which have brought television out of the laboratory and made it a practical device for the use of the general public.

"I present Vladimir K. Zworykin to receive a Howard N. Potts Medal, upon citation reading:

"'In recognition of his outstanding contributions to the science and art of television, which he has advanced not only through the invention of the Iconoscope in which the principle of storage and integration of image light was first employed, but also through the development of the accessory equipment, including the Kinescope, needed in complete commercial television.'"

MR. NALLE: "Dr. Zworykin, by virtue of my office, I present to you the Howard Potts Medal and the Certificate and Report which accompany it."

PRESENTATION OF LEVY MEDALS.

(To Everett McMullin Barber, Beacon Research Laboratory, Texas Company, Beacon, New York; Jay Brent Malin, Beacon Research Laboratory, and Joseph John Mikita, formerly Associate Director of Research, Refining Department, Texas Company.)

MR. NALLE: "I recognize Mr. E. W. Boehne, their Sponsor."

MR. BOEHNE: "Mr. President, I introduce Everett McMullin Barber, Jay Brent Malin and Joseph John Mikita, of Research Departments of the Texas Company; to each of whom has been awarded a Levy Medal for a joint paper by them published in the JOURNAL OF THE FRANKLIN INSTITUTE for April, 1946.

"Their paper, 'The Elimination of Combustion Knock,' embodies outstanding contributions to the automotive industry. Contrary to the layman's general impression that combustion knock is largely controlled by fuel quality, the authors of the above paper reveal that engine design, methods of fuel injection and fuel ignition are of tremendous importance to the problem of combustion knock. The theoretical side of their studies is confirmed by a thorough experimental investigation which also shows the importance of combustible mixture residence times. The part played by fuel Octane and Centane Numbers is particularly well explained in the light of the new data obtained. The experimenters have been particularly ingenious in their method of separating empirically the part played by fuel quality, engine design, and ignition procedures. There is little question that their study will contribute in a substantial manner to the progress of the automotive arts.

"I present Everett McMullin Barber, Jay Brent Malin and Joseph John Mikita, each to receive a Louis Edward Levy Medal, upon citation reading:

"'In recognition of their outstanding paper, "The Elimination of Combustion Knock," appearing in the April 1946 issue of the Journal of The Franklin Institute.'"

MR. NALLE: "Mr. Barber, by virtue of my office, I present to you a Louis E. Levy Medal and the Certificate and Report which accompany it.

"Mr. Malin, likewise, by virtue of my office, I present to you a Louis E. Levy Medal and the Certificate and Report which accompany it.

"Mr. Mikita, in the same manner, by virtue of my office, I present to you a Louis E. Levy Medal and the Certificate and Report which accompany it."

PRESENTATION OF HENDERSON MEDALS.

(To Lars Olai Grondahl, Union Switch & Signal Company, Swissvale, Pennsylvania, and Sedgwick North Wight, General Railway Signal Company, Rochester, New York.)

MR. NALLE: "I call upon Mr. J. V. B. Duer, Sponsor."

MR. DUER: "Mr. President, I introduce Lars Olai Grondahl, Director, Research and Engineering, the Union Switch and Signal Company.

"For many years the possibility of developing a satisfactory means of communication between railroad employees at the front and rear of moving trains and other employees at stationary locations, has been the subject of intensive study.

"Our Medalist made great contributions to the development of such a system known as the Inductive Train Communication system, now in regular service including an extensive main line installation.

"I present Lars Olai Grondahl, to receive a George R. Henderson Medal, which is for distinguished work in railroad engineering; awarded to him, upon citation reading:

"'In consideration of his contributions over a period of years to the development which has resulted in making available a practical system of inductive train communication.'"

MR. NALLE: "Dr. Grondahl, by virtue of my office, I present to you the George R. Henderson Medal together with the Certificate and Report which accompany it.

"Again, I call upon Mr. Duer."

MR. DUER: "Mr. President, I introduce Sedgwick North Wight, of General Railway Signal Company.

"Signaling is an important branch of railroad operation which not only provides means of conveying information to crews of trains as to the speeds at which they are to run and the condition of the track ahead, but also provides for the control and operation of track switches and signals so that safe routes are provided for train movements. Its objective is the safe and prompt movement of trains with a minimum of operational stops.

"Our Medalist has been outstanding in his work in this field and has been granted over 60 United States patents, including the invention and development of major signaling systems known as Absolute Permissive Block and Entrance-Exit Electric Interlocking and inventions and developments relating to Centralized Traffic Control.

"I present Sedgwick North Wight, to also receive a George R. Henderson Medal, upon citation reading:

"'In consideration of his accomplishments in the invention and development of major railroad signaling systems thus contributing to the safety, speed and reliability of railroad operation.'"

MR. NALLE: "Mr. Wight, by virtue of my office, I present to you the George R. Henderson Medal together with the Certificate and Report which accompany it."

PRESENTATION OF WETHERILL MEDAL.

(Kenneth Seymour Moorhead Davidson, Stevens Institute of Technology, Hoboken, New Jersey.)

MR. NALLE: "The Sponsor for the Wetherill Medal, Mr. J. D. Justin."

MR. JUSTIN: "Mr. President, I introduce Kenneth Seymour Moorhead Davidson, of Stevens Institute of Technology.

"In the design of ship hulls, experimentation on models to determine the performance of the ship, the forces to which it will be subjected and its maneuverability are of the greatest importance. From such model tests the performance of the prototype may be predicted.

"The man whom we are about to honor successfully developed the use of the small model for obtaining quantitative results. In many cases he was able to get substantially as accurate results from small models as from large, thus saving time and money in the design of ships and making a material contribution to the successful conclusion of World War II.

"His investigation of such little understood phenomena as 'Porpoising' and the behavior of ships while making sharp turns, with ingenuous instrumentation for measuring and segregating the forces to which a ship's hull is subjected, has made him internationally famous in the field of model testing.

"I present Kenneth Seymour Moorhead Davidson to receive a John Price Wetherill Medal, upon citation reading:

"'In consideration of his contribution to the improved design of ships his experimental work on ship hull models.'"

MR. NALLE: "Professor Davidson, by virtue of my office, I present you with the John Price Wetherill Medal, and the Certificate and Report which accompany it."

PRESENTATION OF CLARK MEDAL.

(Edward George Boyer, Philadelphia Electric Company, Philadelphia, Pennsylvania.)

MR. NALLE: "Mr. W. H. Fulweiler, Sponsor for the Walton Clark Medal."

MR. FULWEILER: "Mr President, I introduce Edward G. Boyer, Manager of Gas Department of the Philadelphia Electric Company.

"Few industries have had to surmount greater technical difficulties during the past three decades than the gas industry. The increase in competition from petroleum, electricity and the liquefied petroleum gases, together with the great increase in the cost of raw materials and the more exacting requirements of the present gas burning appliances, have been a challenge to the technical men in the industry.

"They are meeting this challenge and our candidate is one whose work has contributed in large measure to the solution of some of the technical problems that have beset his industry.

"I present Edward G. Boyer to receive a Walton Clark Medal, which is awarded for distinguished work in the field of manufactured gas, upon citation reading:

"'In consideration of his notable improvements in the development of processes for reforming hydrocarbon gases and for his improved and simplified methods for calculating the required composition of interchangeable gas mixtures and for his other valuable contributions to the art.'"

MR. NALLE: "Mr. Boyer, by virtue of my office, I present to you the Walton Clark Medal, and the Certificate and Report which accompany it."

PRESENTATION OF BROWN MEDAL.

(Karl P. Billner, Vacuum Concrete, Inc., Philadelphia, Pennsylvania.)

MR. NALLE: "Mr. C. Sellers, 3rd, Sponsor for the Frank P. Brown Medal."

MR. SELLERS: "Mr. President, I introduce Karl P. Billner, President of Vacuum Concrete, Incorporated, of Philadelphia.

"In mixing concrete it is necessary to add sufficient water to the dry mix to make it plastic and workable; considerably more than sufficient for complete hydration of the cement.

"Our Medalist has developed a process and apparatus which sucks the excess water from freshly laid concrete by a vacuum system, which also uses the resulting atmospheric pressure to compress the mass and eliminate voids. This double action greatly decreases the time of setting and increases the strength of the concrete.

"Because of these advantages the process has been widely used in many kinds of construction, including highways, Government offices, industrial buildings, housing projects, dam spillways, in this country and abroad.

"I present Karl P. Billner to receive a Brown Medal, which is for distinguished service to the building and allied industries, awarded to him upon citation reading:

"'In consideration of his invention of a practical process for treating concrete, together with the development of suitable equipment whereby the qualities of the concrete are improved and the elapsed time of construction is reduced.'"

MR. NALLE: "Mr. Billner, by virtue of my office, I present to you the Frank P. Brown Medal with the Certificate and Report which accompany it."

PRESENTATION OF CLAMER MEDAL.

(Edgar H. Dix, Jr., Aluminum Company of America, New Kensington, Pennsylvania.)

MR. NALLE: "The next Sponsor is Mr. W. B. Coleman."

MR. COLEMAN: "Mr. President, I introduce Edgar H. Dix, Jr., of the Research Laboratories of the Aluminum Company of America.

"Our Medalist has presented many able papers before technical societies and a number of patents have been issued to him in connection with others associated with him.

"However, a patent in his name alone on the Manufacture of Corrosion-Resisting Aluminum Alloy and Method of Making Same is of especial importance.

"The metal produced under this patent is known as 'Alclad.' It is a pure aluminum covering over a strong aluminum alloy; a combination of qualities meeting requirements for many uses in which aluminum had been failing.

"Military airplanes and many civilian airplanes have been almost entirely constructed of 'Alclad.'

"Without this material, plane construction for our aerial warfare would have been faced with much greater difficulties.

"I present Edgar H. Dix, Jr., to receive a Clamer Medal awarded to him, upon citation reading:

"'For his meritorious contribution to the development of high strength corrosion-resistant aluminum products.'"

MR. NALLE: "Mr. Dix, by virtue of my office, I present to you the Francis J. Clamer Medal together with the Certificate and Report which accompany it."

PRESENTATION OF NEWCOMEN MEDAL.

(Everett G. Ackart, Consulting Engineer, Retired, Wilmington, Delaware.)

MR. NALLE: "I call upon Mr. M. M. Price, Sponsor for the Newcomen Medal."

MR. PRICE: "Mr. President, I introduce Everett G. Ackart, from 1927 to 1946 Chief Engineer of E. I. duPont de Nemours & Co., now retired.

"He has been an active factor in the engineering of that Company since 1907. During that period of forty years he has rendered signal service to this country by his design and construction of powder plants in World War I; by his resourcefulness and pioneering in the design and construction of industrial power plants generating electrical current and supplying steam for industrial purposes; and above all by his masterful leadership in the design, construction and completion of munition plants for the service of the United States during World War II.

"I present Everett G. Ackart for a Newcomen Medal, for achievement in the field of steam engineering, awarded to him upon the following citation:

"'In consideration of his long and successful career of forty years, in designing industrial steam plants and his outstanding accomplishment in designing and building a number of these for the manufacture of smokeless powder at a time when no such plants existed and when the national emergency was great, and in the light of his great accomplishment in building, in time of peace, other installations for the generation of power and steam for industrial use.'"

MR. NALLE: "Mr. Ackart, by virtue of my office, I present to you the Newcomen Medal and the Certificate and Report which accompany it."

PRESENTATION OF BALLANTINE MEDAL.

(George Clark Southworth, Bell Telephone Laboratories, Red Bank, New Jersey.)

MR. NALLE: "Our next award is the Ballantine Medal. It is awarded for the first time tonight. It was founded by friends and associated in memory of Stuart Ballantine, a distinguished radio engineer, a long time active member of The Franklin Institute and a valued member of our Committee on Science and the Arts. I now recognize Mr. Forstall. Mr. W. G. Ellis was to have been the Sponsor for the Ballantine Medal but unfortunately he was unable to be present this evening. Mr. E. L. Forstall has very kindly consented to act in his behalf."

MR. FORSTALL: "Mr. President, I introduce George C. Southworth, Waveguide Research Engineer of the Bell Telephone Company Laboratories."

Continuing, Mr. Forstall had the following presentation remarks prepared by Mr. Ellis:

"It is a particularly personal pleasure to me to introduce our Medalist as the recipient of the first award of our recently established Stuart

Ballantine Medal; to be given for distinguished research in communication and reconnaissance.

"Stuart Ballantine was my close friend from youth; association with him constitutes one of my most valued experiences. He frequently made reference to the stimulation of his ambition received, as a boy, from having been taken to hear outstanding scientists in the old Hall of The Franklin Institute.

"He was long an active member, and in particular of the Science and Arts Committee. It is distinctly appropriate that this Medal has been established in his memory, by a group who enjoyed the inspiration of his friendship, and in the field of endeavor in which his greatest achievements were realized.

"I present George Clark Southworth to receive this honor, 'In consideration of his pioneer work in electromagnetic microwave technique, a material contribution to the development of new systems of communication and reconnaissance radar.'"

MR. NALLE: "Dr. Southworth, by virtue of my office, I take much pleasure in presenting to you the Stuart Ballantine Medal and the Certificate and Report which accompany it."

PRESENTATION OF THE FRANKLIN MEDALS.

(Enrico Fermi, Institute for Nuclear Studies, The University of Chicago, Chicago, Illinois, and Robert Robinson, Oxford University, England.)

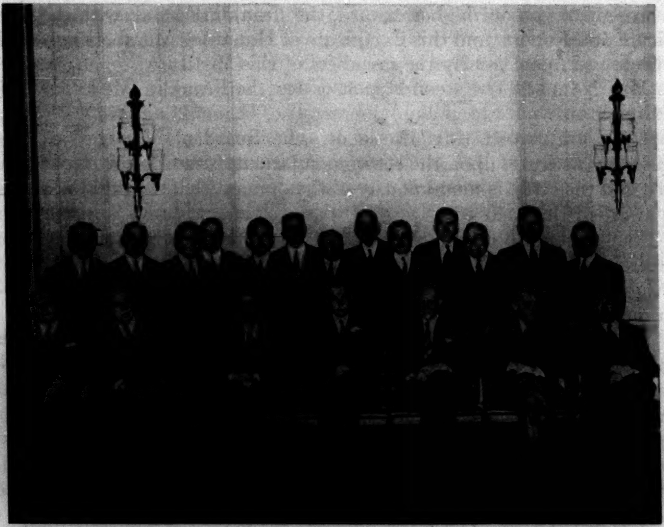
MR. NALLE: "I recognize Dr. F. Palmer as the Sponsor for the first award of the Franklin Medal."

DR. PALMER: "Mr. President, I have the honor to present as a candidate for the Franklin Medal Enrico Fermi, Professor of Physics, University of Chicago.

"In succession a student at the University of Pisa, Lecturer in Physics at the University of Florence, and Professor of Thoeretical Physics at the University of Rome, Professor Fermi came to this country in 1938 and shortly thereafter was appointed to his present professorship.

"His contributions to theoretical physics include (1) the development, at a very early age, of a new system of statistical mechanics, now known as the Fermi-Dirac statistics, which has been the key to the modern theory of metals, and (2) an explanation of the puzzling phenomena observed in the disintegration of radioactive substances by the emission of beta rays. This theory has stimulated further research the end of which is not yet in sight.

"In the realm of experimental physics the name of Fermi is connected primarily with the physics of the neutron. By bombarding the nuclei of the atoms of different chemical elements with slow neutrons he and his collaborators brought about many artificial transmutations of one element into another. It was his attempt, by this means, to create new elements from the heaviest element then known, uranium, which ultimately led Hahn and Strassmann to their epoch-making discovery of the phenomenon of fission, whereby a neutron is captured by the nucleus of a uranium atom which then splits in two and releases other neutrons capable of repeating this process. This is known as a chain reaction.



RECIPIENTS OF AWARDS FOR 1947.

Seated (left to right): Edward G. Boyer (Clark); Harold J. W. Fay (Longstreth); Enrico Fermi (Franklin); Richard T. Nalle, President; Robert Robinson (Franklin); Robert H. Kent (Potts); Vladimir K. Zworykin (Potts).

Standing (left to right): Joseph Razek (Certificate of Merit); Everett G. Ackart (Newcomen); Karl P. Billner (Brown); George C. Southworth (Ballantine); J. Brent Malin (Levy); Peter J. Mulder (Certificate of Merit); Edgar H. Dix, Jr. (Clamer); Sedgwick N. Wight (Henderson); Kenneth S. M. Davidson (Wetherill); Joseph J. Mikita (Levy); Everett M. Barber (Levy); Lars O. Grondahl (Henderson); Samuel Berman (Longstreth).

"The supervision which you, Sir, gave to the design and the construction of the first nuclear chain-reacting pile must be regarded as an engineering accomplishment of the highest order, but it is only one of many vital contributions made by you to the atomic energy project.

"Hence, Mr. President, in recognition of his brilliant contributions to both theoretical and experimental physics, and of his skill in combining theory with experiment so as to open up a vast new field of research in which practical use may be made of the energy stored in the nuclei of certain kinds of very heavy atoms, I take pleasure in presenting Enrico Fermi."

MR. NALLE: "Dr. Fermi, by virtue of the power vested in me as President of The Franklin Institute, I have the honor and the pleasure to present to you our highest award, the Franklin Medal, with the Certificate and Report; and this Certificate of Honorary Membership which is bestowed upon you by the members of this Institute."

MR. NALLE: "The second Sponsor for the Franklin Medal, Dr. H. J. Creighton."

DR. CREIGHTON: "Mr. President, The Board of Managers of The Franklin Institute, upon the recommendation of its Committee on Science and the Arts, is unanimously of the opinion that a Franklin Medal be awarded this year to

"Sir Robert Robinson—'in recognition of his numerous and brilliant researches in organic chemistry, particularly, his invaluable contributions to the knowledge of natural substances, and of his superb experimental skill in organic synthesis.'

"Sir Robert was born in Chesterfield, England, in 1886. After his graduation from the University of Manchester, he was appointed Professor of Organic Chemistry in the University of Sydney, N. S. W., which post he held until 1915. Following his return to England, he occupied successively the chair of chemistry at the Universities of Liverpool, St. Andrews, Manchester, and University College, London. Since 1930, he has been a Fellow of Magdalen College and Waynflete Professor of Chemistry, Oxford University.

"Sir Robert is a Fellow of the Royal Society, and is its present President. He is a Fellow of the Royal Society of Edinburgh and of the Chemical Society, and was the President of the latter from 1939 to 1941. He is a member of many other scientific societies and academies.

"Sir Robert is the recipient of many honors. He was created a Knight in 1939, in recognition of his diversified and important contributions in the field of organic chemistry. He has been awarded honorary degrees by the Universities of Victoria, London, Liverpool, Wales, Dunelm, Birmingham, Edinburgh and Madrid. The Chemical Society has honored him with its Longstaff Medal; the Swiss Chemical Society with its Paracelsus Medal; and the Royal Society with its Davy, Royal and Copley Medals; and in July of this year, at the Centenary Celebrations of the Chemical Society, he is to be awarded the Faraday Medal.

"Sir Robert is generally regarded as one of the world's leaders in organic chemistry, and one of the greatest and most versatile investigators in this field. In his investigations he has shown not only brilliance of conception, but also genius for the selection of methods and remarkable technical resource. In his teaching he has inspired a long and notable succession of students and collaborators by both the fertility of his ideas and his superb experimental skill.

"Sir Robert's researches have covered a signally wide range of problems which have been concerned, primarily, with the chemistry of the life processes of plants and animals, and the results of these investigations have contributed not only to the understanding of the origin and role of natural substances but to an insight into the fundamental mechanism of living things. His long and extensive series of investigations on the constitution and relationships of the plant alkaloids have yielded results which are fundamental to the understanding of their structure and the mechanism of their formation in Nature. Of at least tantamount importance is his work on the anthocyanin and anthoxanthin pigments of plants, culminating in the synthesis of the actual coloring matters of flowers, fruits, and berries. This work, which is regarded as one of the most outstanding achievements in the whole range of modern organic chemistry, has been of special value in the elucidation of the structure of these substances and has excited the keenest interest of chemists and biologists. In recent years, Sir Robert has synthesized important new substances which have effects similar to, but more powerful than, those of natural sex hormones, and which are of great interest for chemotherapy.

"Mr. President, it is a great pleasure and honor to present Sir Robert Robinson."

MR. NALLE: "Sir Robert, by virtue of the power vested in me as President of The Franklin Institute, I have the honor, and the pleasure, to present to you our highest award, the Franklin Medal, with the Certificate and Report; also this Certificate of Honorary Membership which is bestowed upon you by the members of this Institute."

Upon the conclusion of each award, the recipient briefly expressed his appreciation.

MR. NALLE: "Will the Sponsors kindly resume their seats at their respective tables?"

MR. NALLE: "It now gives us great pleasure and we deem it a privilege to have the opportunity to hear from Sir Robert Robinson about Penicillin Chemistry, entitled 'Synthesis of Penicillin.'* Sir Robert."

Mr. Nalle, after Sir Robert had read his paper, conveyed to Sir Robert the thanks of the Institute.

MR. NALLE: "I now present to you Dr. Enrico Fermi who will honor us with a talk on 'The Future of Nuclear Physics.'"†

The President than thanked Dr. Fermi for his paper and dismissed the assembly with a cordial word and declared the meeting adjourned.

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* See page 5 for "Synthesis of Penicillin."

† See page 1 for "The Future of Nuclear Physics."

NEW MEMBERS OF THE FRANKLIN INSTITUTE ELECTED DECEMBER 1, 1946 TO JUNE 1, 1947

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Vannevar Bush, D.E., LL.D., D.Sc.
Enrico Fermi, Ph.D.
Robert Robinson, D.Sc., LL.D.

ACTIVE LIFE MEMBER.

F. O. Laquer, M.D.

SUSTAINING MEMBERS.

Colonel E. H. Molthan
Mr. I. M. Scott
Mr. Howard G. Zeyher

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Mr. Elmer L. Bitner
Mr. Leland H. Burt
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Mr. Julius Dorfman
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Mr. James E. Griffith
Mr. Robert H. Hood
Mr. Henry M. Justi, Jr.
Mr. J. Stanley Kirk
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E. Paul Kitchin, M.D.
Mr. Lycurgus Laskaris
Mr. Bernard N. Levene
Mr. Elmer W. Leyrer
Mr. W. Berkeley MacKenney, Jr.
Mr. Rubin Miller
Mr. William T. Mitman
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Mr. Nathan Widawski
Mr. M. L. Wurman

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Mr. Sebastian Bauer
Mr. Burton David Baylis
Mr. Jack Berger
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Mr. Samuel J. Branco
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Mr. Frank Dunbar
Mr. Edwin B. Dunn
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Mr. Sedgwick N. Wight
Mr. Bruce B. Young

NOTES FROM THE BIOCHEMICAL RESEARCH FOUNDATION.

Seminar on **The Determination of Molecular Weight by Light Scattering.**—MR. W. H. MARKWOOD, JR., of the Hercules Experiment Station, Wilmington, Del., conducted a seminar before the staff of the Biochemical Research Foundation on April 25, 1947, on "The Determination of Molecular Weight by Light Scattering."

Mr. Markwood pointed out that although interest in the determination of molecular size by light scattering has greatly increased in the past few years, the subject is a very old one. The speculations of DaVinci, Clausius, Spring, Lallemand Hagenbach and Nichols as to the nature of the blue of the sky were briefly mentioned. Tyndall's observations of the polarized light scattered from the particles of precipitated vapors were then reviewed.

Strutt's (Lord Rayleigh) 1871 analysis, based on the elastic solid theory, of the dependence of the Tyndall effect upon wavelength, angle and optical density was outlined and his interest in the subject followed through 1881 and 1899. It was recalled that as early as 1873 Lord Rayleigh suspected the scattering to be dependent on particle shape and that the sky color was molecular scattering, not large, suspended particle scattering. His application in 1899 of Maxwell's electromagnetic concept of light was shown to lead to an expression for the "turbidity" (extinction coefficient) of a gas

$$\tau = \frac{32\pi^3(\mu - 1)^2}{3\lambda^4 n} \qquad (1)$$

(wherein μ = refractive index, λ = wavelength, n = number of molecules per unit volume).

The contributions of Mie, Smoluchowski and Debye to the electromagnetic theory of scattering (1908–1909) were mentioned as well as King's (1912) demonstration that sky light was molecular scattering.

Einstein's concept (1910) of the production of scattering in liquids by local, thermal fluctuations of density due to molecular motion was described along with his expression for the turbidity of a pure liquid developed with the help of Boltzman's probability concept of entropy.

$$\tau = \frac{32\pi^3(\mu - \mu_0)^2\mu_0^2}{3\lambda^4}\frac{kT}{\beta(p - p_0)}, \qquad (2)$$

wherein the thermal energy kT is compared with the outside work $p - p_0$ required to produce a density change resulting in a refractive index change of $\mu - \mu_0$.

It was explained that in a somewhat similar manner Debye (1944) showed that in the case of solutions, if p were considered osmotic and depended on kT and n (number of solute molecules), the part of the turbidity of a solution that was caused by the solute could be expressed by

$$\tau = \frac{32\pi^3(\mu - \mu_0)^2\mu_0^2}{3\lambda^4 n}, \tag{3}$$

where $\mu - \mu_0$ is the difference of refractive index between solution and solvent. It was shown that n could be expressed in terms of concentration, molecular weight, osmotic pressure, the gas constant and temperature, resulting in

$$\tau = \frac{32\pi^3(d\mu/dc)^2\mu_0^2 c}{3\lambda^4 N \dfrac{\partial}{\partial c}\left(\dfrac{P}{RT}\right)} = \frac{32\pi^3(d\mu/dc)^2\mu_0^2}{3\lambda^4 N}\,\frac{c}{\left(\dfrac{1}{RT}\right)(dp/dc)} = H\,\frac{RTc}{(dp/dc)}, \tag{4}$$

an equation also arrived at by Mark and Doty in a somewhat different manner.

Since it is known that osmotic pressure is dependent on concentration according to

$$p = \frac{RTc}{M} + Bc^2,$$

$$\frac{Hc}{\tau} = \frac{1}{M} + \frac{2Bc}{RT}. \tag{5}$$

After explaining that (5) was applicable only to solutes whose dimensions were so small compared to λ that the angular dependence of scattering was expressed by $(1 + \cos^2\theta)$, the manner of use of Debye's equations for large molecules to correct (5) was described.

$$I \text{ (small molecules)} = (1 + \cos^2\theta)I_0,$$

$$I \text{ (large spherical molecules)} = \left[\frac{3}{u^3}(\sin u - u\cos u)\right]^2(1 + \cos^2\theta)I_0,$$

$$I \text{ (large coiled molecules)} = \frac{2}{v^2}[e^{-v} - (1 - v)](1 + \cos^2\theta)I_0,$$

$$I \text{ (large rodlike molecules)} = \left[\frac{1}{x}\int_0^{2x}\frac{\sin x\,dx}{x} - \left(\frac{\sin x}{x}\right)^2\right](1 + \cos^2\theta)I_0,$$

wherein I = intensity of scattering observed at an angle θ,
I_0 = intensity of scattering in direction of incident beam,
$u = \pi sd/\lambda$,
$s = 2 \sin(\theta/2)$,
d = diameter of sphere,
$v = (2/3)(\pi rs/\lambda)^2$,

r = R.M.S. separation of ends of random coil,
$x = \pi sL/\lambda$,
L = length of rod.

Also discussed were corrections due to depolarization, reflection from cell walls and absorption.

An equation due to Doty and Mark involving additional terms to (5) was shown to describe the scattering from a solute having a distribution of molecular weights, whereas (5) applies to single species. The additional terms involve the second and third moments $(\Delta M/\bar{M})^2$ and $(\Delta M/\bar{M})^3$ which describe the dispersion of molecular weights about the weight average molecular weight $\bar{M}$.

Various types of scattering apparatus and the preparation of solutions were briefly discussed.